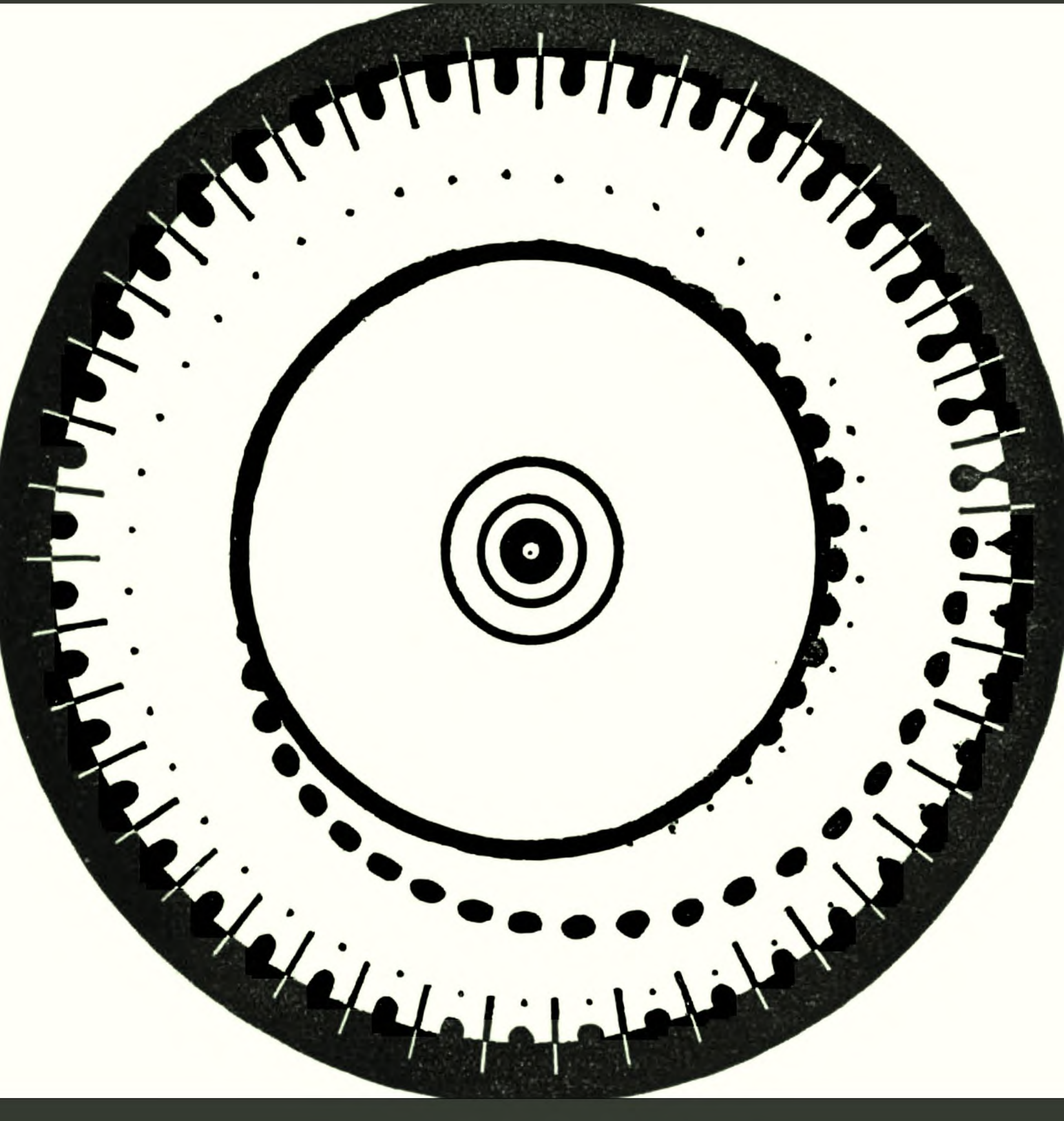
This is a reproduction of a library book that was digitized by Google as part of an ongoing effort to preserve the information in books and make it universally accessible.

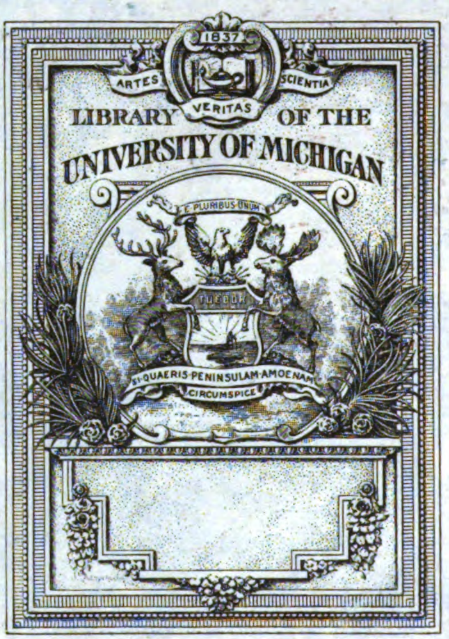
Google books

https://books.google.com



Proceedings of the Physical Society

Institute of Physics and the Physical Society,
Physical Society (Great Britain)


1837
ARTES VERITAS SCIENTIA
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
E PLURIBUS UNUM
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QC
1
.P5821

PROCEEDINGS

44470

OF THE

PHYSICAL SOCIETY OF LONDON.

From June 1890 *to June* 1892.

VOL. XI.

LONDON:
PRINTED BY TAYLOR AND FRANCIS,
RED LION COURT, FLEET STREET.
MDCCCXCII.

ALERE FLAMMAM.

PRINTED BY TAYLOR AND FRANCIS,
RED LION COURT, FLEET STREET.

CONTENTS.

VOL. XI.

Page

Notes on Photographs of Rapidly Moving Objects, and on the Oscillating Electric Spark. By C. V. Boys, A.R.S.M., F.R.S., Assistant Professor of Physics at the Normal School of Science and the Royal School of Mines, South Kensington. (Plate I.) 1

A Formula for Calculating approximately the Self-induction of a Coil. By Prof. John Perry, D.Sc., F.R.S............... 15

On the Measurement of Electromagnetic Radiation. By C. V. Boys, A.R.S.M., F.R.S., Assistant Professor of Physics at the Royal College of Science, London, A. E. Briscoe, A.R.C.S., and W. Watson, A.R.C.S., B.Sc. (Plate II.)............ 20

On certain Relations existing amongst the Refractive Indices of some of the Chemical Elements. By the Rev. Pelham Dale, M.A. (Abstract.) 44

Additional Notes on Secondary Batteries. By Dr. J. H. Gladstone, F.R.S., and W. Hibbert, F.I.C. 44

A Lecture Experiment illustrating the Effect of Heat upon the Magnetic Susceptibility of Nickel. By Shelford Bidwell, M.A., F.R.S.. 47

Alternate Current-Condensers. By J. Swinburne 49

The Solution of a Geometrical Problem in Magnetism. By T. H. Blakesley, M.A. 56

Some Experiments with Selenium Cells. By Shelford Bidwell, M.A., F.R.S.. 61

Experiments in Photoelectricity. By George M. Minchin, M.A., Professor of Mathematics in the Royal Indian Engineering College, Coopers Hill 67

Page

On the Change in the Absorption-Spectrum of Cobalt Glass produced by Heat. By Sir John Conroy, Bt., M.A., Fellow and Bedford Lecturer of Balliol College, and Millard Lecturer of Trinity College, Oxford 103

Further Contributions to Dynamometry, or the Measurement of Power. By T. H. Blakesley, M.A., M.Inst.C.E. 106

Proof of the Generality of certain Formulæ published for a Special Case by Mr. Blakesley. By Prof. W. E. Ayrton, F.R.S., and J. F. Taylor 114

On the Variation of Surface-Tension with Temperature. By Prof. A. L. Selby, M.A., University College, Cardiff 119

The Electrometer as a Wattmeter. By J. Swinburne 122

Alternating and Experimental Influence-Machine. By Mr. James Wimshurst, Member of Council 125

Some Points in Electrolysis. By J. Swinburne 130

The Theory of Dissociation into Ions, and its Consequences. By Spencer Umfreville Pickering, M.A., F.R.S. 139

An Apparatus for Measuring the Compressibility of Liquids. By S. Skinner, M.A., Demonstrator at the Cavendish Laboratory, Cambridge 147

Note on Kohlrausch's Theory of Ionic Velocity. By W. C. Dampier Whetham, B.A., Coutts Trotter Student of Trinity College, Cambridge 149

A Steam-Engine Indicator for High Speeds. By Prof. John Perry, F.R.S. 151

On the Value of some Mercury Resistance Standards. By R. T. Glazebrook, M.A., F.R.S., Fellow of Trinity College, Cambridge 159

Mr. Blakesley's Method of Measuring Power in Transformers. By Prof. J. Perry, F.R.S. 164

Alternate Current and Potential Difference Analogies in the Methods of Measuring Power. By Prof. W. E. Ayrton, F.R.S., and W. E. Sumpner, D.Sc. 172

The Expansion of Chlorine by Light as applied to the Measurement of the Intensity of Rays of High Refrangibility. By Dr. A. Richardson, Lecturer on Chemistry, University College, Bristol. (Plates III. & IV.) 185

Page

The Influence of Surface-Loading on the Flexure of Beams. By Prof. C. A. Carus-Wilson. (Plate V.) 194

Table of Zonal Spherical Harmonics, calculated by Messrs. C. E. Holland, P. R. Jones, and C. G. Lamb. With a short Explanation and some Illustrations of its Use by Prof. John Perry, D.Sc., F.R.S. (Plate VI.) 221

On the Generalizations of Van der Waals regarding "Corresponding" Temperatures, Pressures, and Volumes. By Sydney Young, D.Sc., Professor of Chemistry, University College, Bristol. (Plates VII.–IX.) 233

The Construction of Non-Inductive Resistances. By Prof. W. E. Ayrton, F.R.S., and T. Mather. (Plate X.) 269

A Theory concerning the Constitution of Matter. By Charles V. Burton, D.Sc. 275

Struts and Tie-Rods with Lateral Loads. By Prof. John Perry, D.Sc., F.R.S. 290

On a Permanent Magnetic Field. By W. Hibbert, A.I.E.E., F.I.C. 306

Flexure of long Pillars under their own Weight. By Maurice F. FitzGerald 315

Choking Coils. By Prof. John Perry, D.Sc., F.R.S. 319

On the Construction of a Colour Map. By Walter Baily, M.A. (Plate XI.) 323

A Note on the Electromotive Forces of Gold and of Platinum Cells. By E. F. Herroun, Professor of Natural Philosophy in Queen's College, London 332

On an Instrument for Drawing Parabolic Curves. By Richard Inwards, F.R.A.S. 337

A Portable Instrument for Measuring Magnetic Fields. With some Observations on the Strength of the Stray Fields of Dynamos. By Edwin Edser and Herbert Stansfield. (Plates XII. & XIII.) 338

Note on the Measurement of the Internal Resistance of Cells. By E. Wythe Smith 342

Breath Figures. By W. B. Croft, M.A. 346

Inductoscript. By Frederick J. Smith, M.A., Millard Lecturer Mech. et Phys., Trinity College, Oxford 353

Page
On the Relation of the Dimensions of Physical Quantities to Directions in Space. By W. Williams, Assistant in the Physical Laboratory, Royal College of Science 357

Index 399

[At end of Volume.]

Proceedings at the Meetings, and Report of the President and Council for the Session 1890–91.

Proceedings at the Meetings, and Report of the President and Council for the Session 1891–92.

PLATES.

I. Illustrative of Mr. C. V. Boys's Paper on Photographs of Rapidly Moving Objects and on the Oscillating Electric Spark.

II. Illustrative of Messrs. Boys, Briscoe, and Watson's Paper on the Measurement of Electro-magnetic Radiation.

III. & IV. Illustrative of Dr. Richardson's Paper on the Expansion of Chlorine by Light as applied to the Measurement of the Intensity of Rays of High Refrangibility.

V. Illustrative of Prof. C. A. Carus-Wilson's Paper on the Influence of Surface-Loading on the Flexure of Beams.

VI. Illustrative of Prof. J. Perry's Paper on the Table of Zonal Spherical Harmonics calculated by Messrs. Holland, Jones, and Lamb.

VII.-IX. Illustrative of Prof. Sydney Young's Paper on the Generalizations of Van der Waals regarding "Corresponding" Temperatures, Pressures, and Volumes.

X. Illustrative of Prof. Ayrton and Mr. Mather's Paper on the Construction of Non-Inductive Resistances.

XI. Illustrative of Mr. Baily's Paper on the Construction of a Colour Map.

XII. & XIII. Illustrative of Messrs. Edser and Stansfield's Paper on a Portable Instrument for measuring Magnetic Fields.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

NOVEMBER 1890.

I. *Notes on Photographs of Rapidly Moving Objects, and on the Oscillating Electric Spark. By* C. V. BOYS, *A.R.S.M., F.R.S., Assistant Professor of Physics at the Normal School of Science and the Royal School of Mines, South Kensington* *.

[Plate I.]

I HAD occasion last Christmas to show to a juvenile audience a modification by the late Dr. Guthrie of one of Plateau's experiments, which beautifully illustrates the effect of the surface-tension of a liquid during the growth and formation of a drop. Under ordinary circumstances a drop of water is so small that the formation of the neck and separation of the drop are too rapid for it to be possible to follow the process by the eye. If the drops were larger the process would be slower, but a larger drop, owing to the weight of water, cannot be formed in air. Dr. Guthrie caused the drop to grow at the end of a pipe half an inch or so in diameter dipped below the surface of paraffin. By this means the effective weight of the drop is very much reduced, and the surface tension is reduced, so that a large drop can be formed, and the process is made comparatively slow. I have found that the development of the drop can be made still more gradual, and that still larger drops may be formed by adding bisulphide of carbon to the paraffin so as further to reduce the effective weight of the drop.

---

* Read May 2, 1890.

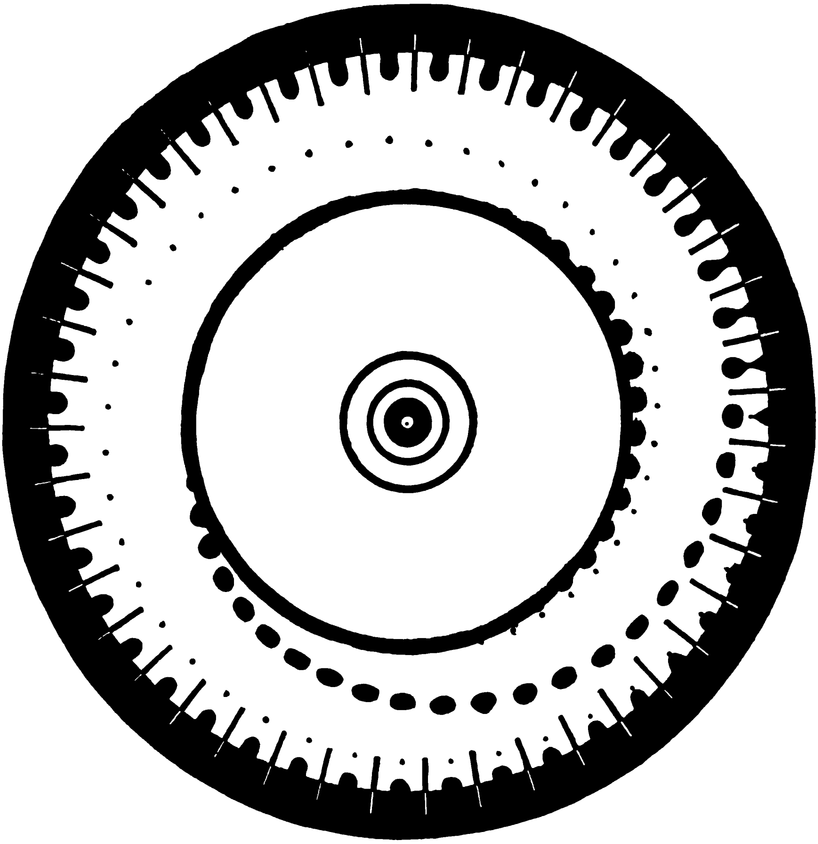
While trying to devise some way of representing this experiment on paper, I saw at the Royal Institution an apparatus exhibited by Mr. Friese Greene, with which photographs at the rate of ten a second could be taken upon a travelling film moved on between and stopped during each exposure.

Examination of the growing drop through a small window in a card disk made to rotate ten times a second, showed that even this rate applied to large and slowly forming drops was not sufficient to give anything like a continuous representation of the forms assumed. However, I found no difficulty in obtaining twenty photographs a second; and, had I required them, I could certainly have obtained fifty sharp and fully exposed photographs in this time, by the following method:—An electric arc was focused by a pair of seven-inch lenses upon the lens of an ordinary half-plate camera. A box with a glass front and back, containing the liquids and the growing drop, was placed immediately in front of the large lenses, and the drops were focused by the lens of the camera upon the ground-glass screen at the end of a slide forty-two inches long. This could be charged with four 10 × 4 plates (two 10 × 8 plates cut longitudinally), and then be drawn by hand from one end to the other of a seven-foot slide which protected the plates from light everywhere except at one place where a window 4 × 1 inches had been cut. Through this window the image of the falling drops was cast upon the moving plate. Just in front of the camera-lens a disk of card with an aperture near the edge was made to rotate at any desired speed by one of Cuttriss's P. 1 motors. Everthing was then started, and the speed of the motor and rate of dropping adjusted until the really intermittent but apparently continuous view of the phenomenon upon the ground plate was as desired. The slide was then rapidly drawn from end to end by hand, the speed being such as to obtain rather more than one complete cycle during the motion of the slide. In the photograph, of a portion of which Plate I. fig. 1 is an exact copy, the exposures were at the rate of fourteen a second, and lasted about one five hundred and sixtieth of a second each. The exposure was more than enough to make a perfectly black and white negative, and the outlines were clear and sharp. Another series taken at the rate of twenty a second with an exposure of one eight hundredth of a second each, in which forty-three separate views were taken in the complete cycle, was cut up into strips and mounted on a large disk of cardboard with forty-three equidistant and narrow slits near the edge. When the cardboard is made to rotate in front of a looking-glass, after the fashion of the thaumatrope, the original experiment is again perfectly presented, and all the features are evident without the necessity for using these objectionable liquids. The drop is seen to slowly enlarge until it is too heavy to be supported by the surface-tension, the form becomes unstable, a neck begins to develop, which gradually gets narrower until the drop separates. The free drop being suddenly released from the one-sided pull, vibrates as it falls, not in the manner figured in the text-books, between a prolate and oblate spheroid, but rather in a triangular fashion, with the base and vertex alternately uppermost, as recently described by Ph. Lenard*. The falling drop reaches the water at the bottom of the box, where it vanishes for a time only, for it has not mixed with this water, and so it presently bounces out, as shown in fig. 1, after which it again falls into the water. Meanwhile the end of the neck from which the drop broke away has gathered itself together into the little drop or spherule so ably described by Plateau, which has an upward velocity imparted to it, since it separated from the upper or pendant drop after the falling drop had broken away. Moreover, just as the falling drop vibrates after it is released, so the pendant drop vibrates also and grows as it vibrates, coming into collision with the upward moving spherule, which it drives away by a blow. Fig. 2 is a reduced copy of the thaumatrope. The method described could be made more perfect by using films on rollers driven by the same motor that makes the card rotate, so that the separate photographs should neither overlap nor leave space unoccupied; but I did not make use of this refinement, as my object was simply to get the true outline in a particular experiment, and not to construct a piece of mechanism. The apparatus described was simply extemporized out of the materials at hand, and it answered my purpose perfectly.

---

* *Annalen der Physik und Chimie*, vol. xxx. 1887.

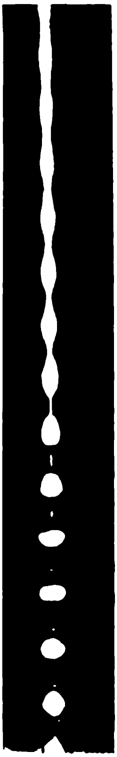
Fig. 2.



2. As I wished in the lectures referred to above to show that exactly the same process goes on even when the finest jet of water breaks up into beads, though of course the process is hundreds or thousands of times more rapid, I was anxious to obtain photographs of these forms which should be as sharp and as clear as the photographs already described. Mr. Chichester Bell described to me in a letter a method which he had used successfully for this purpose, which is more especially interesting now, as Lord Rayleigh has recently

exhibited photographs taken by a very similar method. Mr. Chichester Bell simply makes a small spark with one or two Leyden jars several feet away from the plate. The jet of water is allowed to fall between the spark and the plate, but as close to the plate as possible. The shadow cast under these conditions is perfectly sharp and will stand considerable magnification, and the light is quite sufficient to blacken the plate except when the water-drop protects it. Fig. 3 is an enlargement of one of these photographs. The water was broken up regularly by simply blowing into a key which whistled a note of 2730 complete vibrations a second, so that each drop moved to the position occupied by the next in the $\frac{1}{2730}$ of a second. Lord Rayleigh photographed jets of water by the spark with a camera and lens, using in addition a large condensing and field lens, so as to obtain sufficient light; but it appears that the glass of the lenses is not of much advantage, if any, because, as he himself has since suggested, it absorbs so large a proportion of the ultra-violet rays that the gain by concentration is about balanced by the absorption of the most actinic rays.

Fig. 3.



3. Having found by the shadow method that there was an abundance of light from a comparatively feeble spark, I thought it possible that each half period of an oscillating spark might produce sufficient light to leave a clear record on a rapidly moving photographic plate of the shadow of small and rapidly moving objects. To put this to the test of experiment, the following arrangement was made. In the first place an oscillating spark was formed between a pair of polished brass knobs about one tenth of an inch apart, by discharging a condenser of large capacity through one or two coils of gutta-percha-covered copper wire to give self-induction.

Details of the electrical arrangements will be given at the end of these notes; but I may state here that the separate flashes of light in the spark followed one another at the rate of about 4500 a second. A "half plate" was held in a recess in a disk of wood screwed to a whirling-table, so that it could be made to rotate in its own plane at a high speed. A card-screen with a narrow slit was placed just in front of the plate, so that the light from the spark could fall on the plate through the slit. The jet of water was then formed just in front of the slit, and when the plate was rapidly revolving the spark was formed. The plate, on being developed, was found to have impressed upon it a picture like an open fan with the water-drops arranged in spiral lines. This method was not so convenient as a modification in which, instead of a glass plate spinning in its own plane, a flexible film was employed, wrapped round and held by two india-rubber bands upon a brass cylinder 1⅓ inches in diameter. This could be made to rotate at a very high speed by means of a light band from a whirling-table. Pl. I. fig. 4 is a copy of a photograph obtained in this way. Here it will be seen that the intermittent light is automatically produced, and the period of the intermittence is constant for any particular apparatus. However, as carried out, the value of this method of observing the movements of rapidly moving bodies loses its practical value because the spark is alight during a large proportion of the time of each oscillation, so that it is by no means an instantaneous phenomenon.

4. In order to show the oscillating spark and to examine it by photography, I wished to have some more convenient means than the rotating mirror employed by Dr. Lodge. With this the observer is taken by surprise whenever a spark passes, for he does not know exactly in what direction to look, and therefore he finds it more difficult to recognize the peculiarities of the drawn-out spark than he would if he were certain that he would see the spark in the exact direction in which he was looking. Besides this it is not easy, when photographing the spark, to ensure that the whole phenomenon from the birth to the dying flickers of every spark shall be impressed upon the plate.

The method which I am about to describe appears to me to attain these ends more perfectly than any other with which I am acquainted, and it does not introduce any difficulties of its own.

A brass disk (Pl. I. fig. 5) has six achromatic opera-glass lenses of about seven inches focus screwed into it, but not at the same distance from the axis. One pair, 1, 1, on opposite sides of a diameter are placed equidistant from the axis; the next pair 2, 2 are placed one tenth of an inch nearer the axis; the third pair 3, 3 one tenth of an inch nearer still. The wheel is mounted on an axle, carrying a grooved pulley-wheel, and is accurately balanced. It may then be made to rotate, if necessary, at the very high speed of over a hundred turns a second. The spark is formed on one side of the plate, and a screen or photographic plate is held on the opposite side; and the two are so adjusted that the spark is focused upon the screen or plate. Now, when the disk is made to rotate, the image of the spark is drawn out into a beaded band; but as it is possible to turn the disk so fast that it has moved through an angle of nearly one hundred and eighty degrees during the existence of a spark, while to turn it through a right angle is perfectly easy, each lens will project a beaded arc of a circle perhaps nearly half a turn in extent. If the separate lenses were all at the same distance from the axis, these six semicircular arcs would be projected on the top of one another and confusion would result; but as they are displaced by an amount which is about the same as the length of a convenient spark, the separate arcs are just made to clear one another. Therefore, if the plate is large enough to take an arc of the length of sixty degrees, the whole record of the spark will afterwards be found upon it. By this means it is possible to make use of the high speed which is necessary to widely separate the several constituents of the oscillating spark without at the same time causing the record to be spoiled by the overlapping of the image formed by one lens by that produced by the next. A single pair of lenses, instead of six, would prevent the overlapping, but then to certainly obtain the whole record it would be necessary to make the plate receive an arc of one hundred and eighty degrees, which, on optical grounds, would be objectionable. If the sparks are to be shown on a screen it is best to make the spark in the axis of rotation, and show the complete circle or a large part of it; but if the image is to be photographed, it is best to place the spark exactly opposite one lens and use such a size of plate that when the disk is turned one image just begins to leave the plate on one side when the next is coming in on the other.

In order to make the most of a plate, it is convenient to expose it in a common camera-slide held in the hand or in a groove behind a card screen, with a curved aperture of such a size that the image from all three lenses can fall upon the plate. The plate can then be moved on by hand one stage after every spark, and five or six groups taken on one half plate. Specimens of portions of a few of the photographs obtained are arranged together in Pl. I. fig. 6.

Dr. Lodge has lately photographed oscillatory sparks, and has observed their peculiarities; I shall therefore do no more than merely mention the most striking features. The spark is obviously alight nearly all the time that it lasts, the intervals of darkness are short. The oscillatory spark proper is, I believe, always determined by what may be called a pilot-spark, due to the discharge of the knobs and wires outside the self-induction.

The pilot-spark is well seen at *p* in A, fig. 6, which is a photograph taken with a very high speed and considerable magnification. That this pilot-spark is due to the discharge of the knobs is certain, because in the first place it is a quarter period behind the centre of the first spark of the oscillation proper, and in the second place because in a photograph taken under similar conditions, but after a small Leyden-jar had been directly connected to the terminals, the pilot-spark was far brighter than before and was even brighter than any of the succeeding alternating elements, while it occupied the same place, namely one quarter period before the centre of the first spark of the oscillation proper. The very exact analogy between the motion of the electricity in an oscillating spark and of the bob of a common pendulum may be worth tracing:—

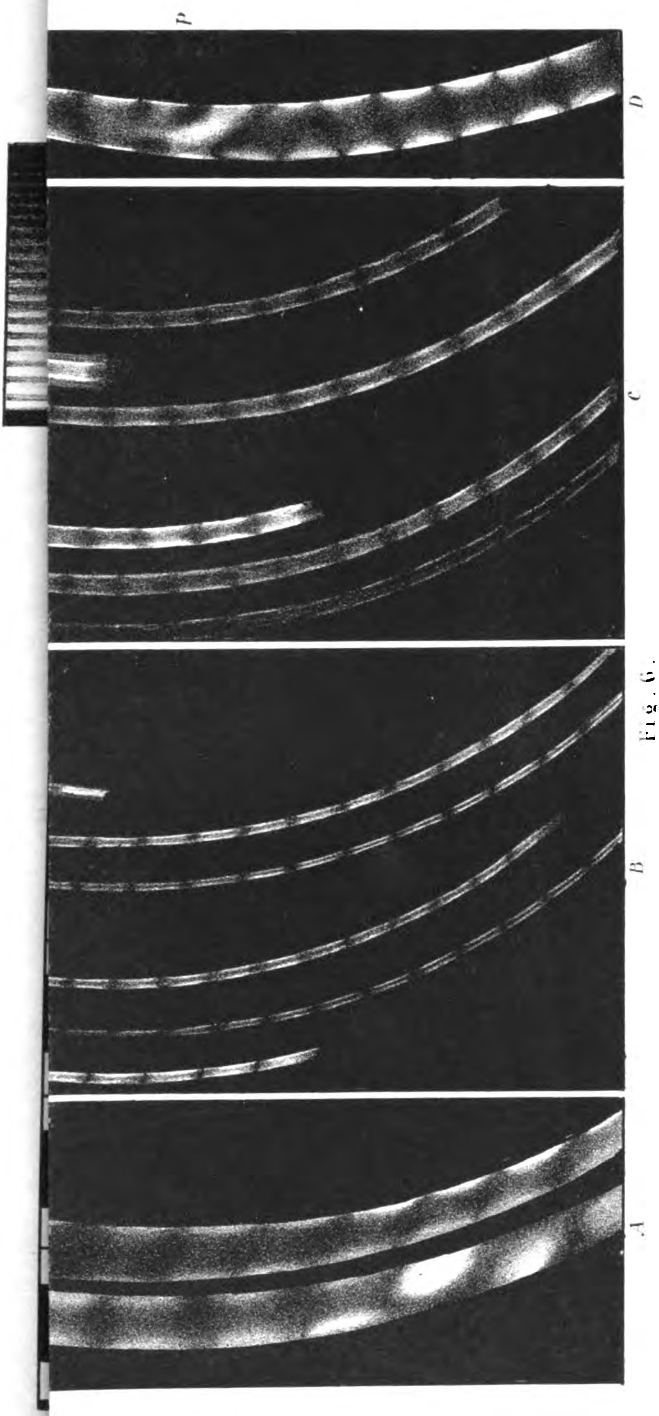


Fig. 6.

The electric displacement gradually increases as the machine is worked until the electromotive force is sufficient to break the air-gap which held the electricity on one side.	The displacement of the bob gradually increases as the thread which draws it to one side is pulled until the force is sufficient to break the thread which held the bob on one side.
The steady oscillation of the electricity then begins, and in a quarter period the electric velocity is at its greatest value, the current is strongest, or the spark is brightest; the current then gradually gets less until for a moment it ceases and the spark goes out.	The steady oscillation of the pendulum then begins, and in a quarter period the velocity is at its greatest value; the velocity then gradually gets less until for a moment the bob is at rest.
The electric displacement is now nearly the same in amount as at first, but in the opposite direction, it therefore starts back again, and so swings backwards and forwards as the knobs become alternately + and − (as is indicated in some of the photographs by the brighter glow alternately at each end of the spark), but getting less at every swing, owing partly to electrical resistance wastefully producing heat and partly to electromagnetic radiation which will give rise to oscillating currents in neighbouring conductors, especially if they have the same period, until at last the whole store of energy has been wasted and the electricity is at rest.	The displacement of the bob is now nearly the same as at first, but in the opposite direction; it therefore starts back again and so swings backwards and forwards, resting alternately on one side and on the other, but getting less at every swing, owing partly to resistance wastefully producing heat and partly to wave-motion transmitted through the supporting wall which will cause other pendulums to start swinging, especially if they have the same period, until at last the whole store of energy has been wasted and the bob is at rest.

In order to increase the total number of oscillations, I have introduced what I call a trap, which consists of a second air-gap in the circuit kept open by a prop while a strong spring tends to close it. The spark-gap is then made very short, but the potential can be raised up to any degree that the condenser will safely stand. A string tied to the prop is then pulled, and the spring completes the second gap so quickly that not more than one or two oscillations can take place after the spark begins and before the filling of the air-gap at the trap. Thus a higher potential can be obtained than would be possible if the short working spark-gap alone determined the potential, while the resistance in circuit is less than before owing to the shortness of the spark.

Two sparks taken with the trap are shown in B, fig. 6. Two longer sparks taken without a trap are reproduced in C, fig. 6. The gain in number of oscillations is not very great, as the measures by Messrs. Robson and Smith at the end of this paper show ; but there is a gain, especially as compared with a short spark taken without a trap. The trap, however, is practically useful in another way, it enables one to fix the instant at which the spark shall occur, which, besides being convenient, is important if for any reason the highest possible speed of rotation is required. While the lenses will run continuously at so low a speed comparatively as fifty or seventy revolutions a second, it would not be well to run them at high speeds for more than a few seconds together. By the use of the trap this is possible, whereas, if the high speed had to be kept up during the long time that it takes to charge a condenser of so great a capacity to the sparking-point, there would be great risk to the bearings of the wheel of lenses. For instance, to obtain a spark about one tenth of an inch long from the condenser used in these experiments, from twenty to thirty turns of the handle of a large eight-plate Wimshurst machine had to be made.

Anomalous variations of brilliancy are not unusual, as may be seen in A, but a curious example is indicated by the letter *p* in D. This represents the latter or dying portion of a spark taken on the same negative and under the same general conditions as A. In this case it will be seen that the light instead of going out at the end of an oscillation travelled slowly across the gap to the other terminal, forming apparently a kind of ball-discharge.

One series of photographs I may here refer to, though the effect sought for is not so evident as I had expected. In this series a short vertical spark was formed, with its length parallel to the direction of the motion of the lens which at any time might be opposite to it. A magnetic field was produced in the line joining the spark and the lens. Then it was hoped that the current oscillating backwards and forwards would be largely deflected laterally alternately on either side so as to produce a zigzag impression. The arrangements were very imperfect, but there is a very slight lateral shift which I believe to be real, but it will be necessary to try the experiment again under more perfect conditions.

I must express my obligation to our lecture-assistant, Mr. R. Chapman, who has performed the whole of the purely photographic part of the experiments, and whose experience in this direction has been of very great service.

The following are the particulars of the apparatus that has been used. The electrical constants were determined by Messrs. W. G. Robson and S. W. J. Smith.

Condenser.

A wooden tray was made, as shown in fig. 7, 28¼ inches long and 16¼ inches wide inside, but with the base projecting on each side so as to form a 22-inch square. A sheet of varnished glass of this size was laid upon the base, and then twenty-six pieces of thin window-glass 28 × 16 inches were piled on one another in the tray. The window-glass had been well washed, dried, and varnished while hot on each side all round the edge for a width of two inches. Pieces of tinfoil 24 × 12 inches were laid on each of the oblong pieces of glass, except the top one, and long tongues of tinfoil 3 inches wide were brought out alternately on the two sides. On one side all these tongues were brought out in the middle and bent down so as to rest on the square glass plate below; on the other side they were brought out in four sets of three each with a space of two inches between each set. These were separately bent down to lie on the square glass plate below. By this means the capacity can be made variable in the proportions of 1, 2, 3, and 4 by simply connecting the wire to 1, 2, 3 or 4 of these electrodes. A strip of tinfoil lying over all the ends and held in contact by weights keeps them all connected except when a smaller capacity is required.

Total capacity 0·0979 microfarad.

Total capacity of battery of ten large jars 0·0451 microfarad.

Total capacity of small jar used to connect knobs (p. 8) ·0026 microfarad.

Self-Induction.

Two coils, containing 330 yards each of No. 16 copper wire covered with gutta-percha to No. 7, were made use of in the form in which they arrived from the makers. The two coils are marked C and U. The following are the dimensions in inches approximately:—

	Outside diam.	Inside diam.	Depth.
C. . .	12	$8\frac{1}{4}$	6
U . .	$11\frac{1}{2}$	$7\frac{1}{2}$	6

The resistances of the coils are:—

	B.A. units.	Ohms.
C	2·423	2·391
U	2·514	2·481

The self-inductions determined by Sumpner's method (Journal of the Soc. of Telegraph Engineers, 1887, fig. 5) were found to be

C . . .	$26 \cdot 89 \times 10^6$	C.G.S. units.
U . . .	$26 \cdot 31 \times 10^6$	„ „

In comparing the number of oscillations measured with those calculated from the electric constants given, experience showed that the photographs, such as those in fig. 6, were not so suitable as one taken with the spark in the line of the axis of the wheel of lenses, because, though the definition along a radius was not so good in the latter arrangement, it was for the purpose of determining the intervals of darkness just as good, moreover it gave a much larger arc on one plate which made it possible to obtain the centre on the plate, and therefore the angle between two points on the arc, with greater accuracy. If the lenses were screwed into the face of a flat cone so that the axes of each should meet in the axis of rotation at the place of the spark, then definition would be equally perfect in all parts of the circle. I did at first intend to place the lenses in such a cone, but for various reasons the plate seemed preferable.

The comparison of the calculated number of oscillations with the number found is complicated by the relations between true ohms and B.A. units, as follows:—

If the condenser is standardized with reference to the B.A. unit of resistance, then, since the B.A. unit is 1·33 per cent. too small, the number expressing the resistance is too large and therefore the number expressing the capacity is too small, since one varies inversely as the other, and so the number expressing the capacity of such a condenser must be multiplied by 1·0133 to obtain the true capacity.

The self-induction determined by comparison with capacity and resistance is found from the expression

$$L = \text{a number} \times r_1 \times r_2 \times C.$$

If the resistances r_1 r_2 are given in true ohms, then, as the number expressing C is too small by 1·33 per cent., therefore the number expressing L is also too small by 1·33 per cent., and the nominal value of L must be multiplied by 1·0133. But if r_1 and r_2 are given in B.A. units, then, since these numbers are each too large by 1·33 per cent., therefore the number expressing L is also too large by this amount, since one of the resistances and the capacity will balance, and therefore the number expressing L must be divided by 1·0133.

Now, the decrement being small, the numbers of oscillations in a second $n = \frac{1}{2\pi\sqrt{LC}}$; therefore, if L and C have each to be multiplied by 1·0133, n should be divided by 1·0133; but, on the other hand, if one has to to be multiplied and the other divided by 1·0133, then these corrections balance and n requires no correction.

Messrs. Robson and Smith determined from the photographs the actual number of oscillations per second, and they determined also, assuming the microfarad to be correct, the number that ought to have been produced. The following are their results :—

Observed number . . .	2293	3170	3264	2115	1806
Calculated number . .	2210	3126	3126	2210	1826
Number of elementary sparks in complete spark	16	23	20	15	14

These were all taken on half plates and were subject to the difficulty of centering. One spark was taken on a large plate, so that all six lenses produced images of the same spark. In this case the observed number was 1852, and the calculated number 1826. Now, if this number is corrected by dividing it by 1·0133, the calculated number becomes 1802, showing the actual number to be about 2·7 per cent. too great. As window-glass was used for the condenser, which, owing to soakage, gives smaller capacities as the time of discharge is made smaller, it is probable that part of the discrepancy may be due to this ; but in the measurement of the capacity of the condenser the effect of soakage was made as small as possible by making, as far as such a thing could be done by hand, the contact of discharge instantaneous. The self-induction, too, as measured, may have been slightly different to the self-induction for oscillations at the rate of 1852 a second.

Mr. Smith has calculated the self-induction, using Prof. Perry's formula, given at the last meeting of the Physical Society, but the irregular form of the coils would not permit of an accurate number being obtained in this way. The figure obtained is about 51×10^6, a value which agrees with the observations better than that found by experiment.

I should add that I do not attach any importance to the discrepancies referred to. The experiments were made mainly with the view of showing the oscillating spark to an audience more clearly than is possible by the method of Dr. Lodge. For the purpose of measurement a window-glass condenser is not suitable at all, and I do not think the rotating lenses are so suitable as the method employed by Dr. Lodge, in which the plate itself is made to rotate in its own plane, for then the centre can be determined with the greatest accuracy.

The actual gain in number of sparks produced by the trap was found by making three experiments, in which all the conditions were the same except those specified.

A. No trap . .	$\frac{1}{10}$ inch spark	32	elementary sparks.
B. No trap . .	$\frac{1}{20}$ inch spark	29	"
C. Trap giving potential of A.	$\frac{1}{20}$ inch spark	37	"

I should in conclusion state that the measurements given at the end of these notes were made after the meeting of the Physical Society at which the paper was read, and that some other additions have been made to the fourth part of these notes.

II. *A Formula for Calculating approximately the Self-induction of a Coil.* *By* Professor JOHN PERRY, *D.Sc., F.R.S.*[*]

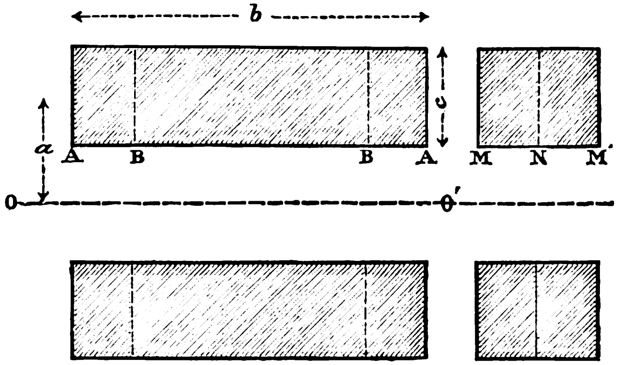
AN easy empirical formula for enabling the coefficient of self-induction (*the inductance*) of any coil to be calculated with approximate correctness is very much wanted. I have been led to take up the matter through having to design an instrument with various cylindric coils whose inductances should bear certain relations to one another, and it was necessary for me to see at a glance with some exactness the effect of altering their dimensions. It is to be observed that for practical purposes extreme accuracy of calculation is of no importance, because the mechanical inaccuracies in constructing coils are considerable. It will be seen that my method of arriving at what may almost be called a rational formula has been suggested by Prof. James Thomson's ingenious method of considering the flow of water over a rectangular gauge-notch.

I consider only hollow cylindric coils. Thus fig. 1 shows the section of a coil whose dimensions are c centimetres measured radially, b parallel to the axis, mean radius of coil a, $o\,o'$ being the axis. The supposition I make is that all the coils considered are so long (b so great) that, $\frac{c}{a}$ being the same in all, the field near the ends is the same in all. Without much straining of the meaning of words, we may say that the inductance of a coil is the total induction when unit current passes in the coil; and if there is only one convolution, it is proportional to the reciprocal of what may be called the magnetic resistance. My supposition comes to this: that if the dimension b is great enough, increase of the magnetic resistance produced by increasing b is proportional to the increase of b. Let M N M′ (fig. 2) show the limiting length beyond which increase of resistance will be proportional to increase of

[*] Read June 20, 1890.

length. Evidently M M′ must be a function of c and a, and if written in the shape $2sc$ it may be expected that s will be constant for small values of $\frac{c}{a}$. The magnetic resistance due, then, to the end parts, A B, of the coil (fig. 1) will be the same as that of the coil, fig. 2. Now the magnetic resistances of all similar coils are inversely proportional to their dimensions, and for all coils similar to that shown in fig. 2 for any value of a or c but for the same values of $\frac{c}{a}$, the magnetic resistance is inversely proportional to a, say that it is $\frac{p}{a}$, p being a function of $\frac{c}{a}$. Now the remaining part of the coil (fig. 1) may be considered made up of coils of square section $c \times c$ and of mean radius a; and the additional magnetic resistance introduced by each of these is inversely proportional to a if $\frac{c}{a}$ is constant. There are $\frac{b-2sc}{c}$ of them, and the total magnetic resistance, irrespective of the ends, is proportional to $\frac{b-2sc}{ca}$ if $\frac{c}{a}$ remains constant. Say that it is $(b-2sc)\frac{q}{a^2}$, where q is a function of $\frac{c}{a}$. The whole magnetic resistance is then

Fig. 1. Fig. 2.



$$\frac{p}{a} + \frac{q}{a^2}(b-2sc).$$

And the inductance L for one winding is the reciprocal of this. If there are n windings,

$$L = \frac{n^2 a^2}{pa + qb - 2sqc}. \quad \ldots\ldots \quad (1)$$

As sc is comparable with the axial distance inside a coil from the end, at which the lines of force are no longer perceptibly curved, s may be supposed to be a proper fraction. In finding the values of p, q, and s, p and q being functions of $\frac{c}{a}$, and s being nearly constant, it is evident that if we, experimentally or by calculation, find L for various values of a, b, and c, it is enough if one of these remain constant. Mr. C. E. Holland has worked out for me the following values of $\frac{L}{4\pi n^2}$ in C.G.S. units, keeping $c=1$ centimetre, from the series published as Appendix III., page 321, vol. ii. of the second edition of Maxwell s 'Electricity and Magnetism.' Unfortunately, unless the ratios of $\frac{c}{a}$ and $\frac{b}{a}$ are small, this series does not enable L to be calculated. I propose later to amend the formula when I have, either by calculation or experiment, found L for values of $\frac{c}{a}$ and $\frac{b}{a}$ greater than $\frac{1}{2}$; and I regret that the difficulty of obtaining values of L in these cases is so great as it is. Calling Mr. Holland's calculated $\frac{L}{4\pi n^2}$ by the letter l, the following Table shows the results of his calculations:—

a.	b.	c.	l.
4	1	1	9·060
4	2	1	7·303
4	3	1	5·993
2	1	1	3·094
2	2	1	1·985
2	3	1	1·034
4	1	2	7·460
3	1	1	5·914
3	2	1	4·510
3	3	1	3·420
6	1	1	16·0436
6	2	1	13·527
6	3	1	11·704
10	1	1	31·8655
10	2	1	27·763
10	3	1	24·834

Now it is evident that if of any coil the dimensions are a, b, and c, then L for this coil is c times the L for a coil whose dimensions are $\frac{a}{c}$, $\frac{b}{c}$, 1. Hence Mr. Holland's numbers enable us to study the general case, so long as $\frac{c}{a}$ and $\frac{b}{a}$ are less than $\frac{1}{2}$. I find that with more than enough accuracy for practical purposes we may take, when $c=1$ centim.,

$$\frac{\mathrm{L}}{4\pi n^2} \text{ or } l = \frac{a^2}{\cdot 2317\,a + \cdot 39 + \cdot 44\,b}. \quad . \quad . \quad . \qquad (2)$$

Hence generally, from the law as to similar coils, l is

$$\frac{c\,\frac{a^2}{c^2}}{\cdot 2317\,\frac{c}{a} + \cdot 39 + \cdot 44\,\frac{b}{c}}, \text{ or } \frac{a^2}{\cdot 2317\,a + \cdot 39\,c + \cdot 44\,b}.$$

So that generally

$$\text{L in centimetres} = \frac{4\pi n^2 a^2}{\cdot 2317\,a + \cdot 39\,c + \cdot 44\,b}. \quad . \qquad (3)$$

I give it in this form to enable its accuracy to be tested. And it is obvious that q of (1) is a constant, whereas p is a linear function of $\frac{c}{a}$ including a constant.

$$\text{L in centimetres} = \frac{n^2 a^2}{\cdot 01844\,a + \cdot 031\,c + \cdot 035\,b}. \quad . \qquad (4)$$

Or taking the ohm as 10^9 centimetres per second,

$$\text{L in secohms} = \frac{n^2 a^2 \div 10^7}{1\cdot 844\,a + 3\cdot 1\,c + 3\cdot 5\,b}. \quad . \quad . \qquad (5)$$

If α is the cross sectional area pertaining to one winding, whether made up partly of insulation or not,

$$n = \frac{bc}{\alpha};$$

and if R is the resistance of the coil in ohms, as $\mathrm{R} = \frac{2\pi a\rho bc}{\alpha^2}$,

$$\frac{\mathrm{L}}{\mathrm{R}} = \frac{\mathrm{V} \div 10^9 \pi\rho}{\cdot 2317\,a + \cdot 39\,c + \cdot 44\,b}; \quad . \quad . \quad . \quad . \qquad (6)$$

where V is the volume of the coil in cubic centimetres, $\rho =$ resistance in ohms of a copper wire 1 centim. long, of smaller section than 1 centim., just as much smaller as the actual wire is than the covered wire.

Let $\rho_0 =$ specific resistance of copper in ohms.

Taking it that if there is no insulation, $\rho = \rho_0 = 1\cdot 62 \times 10^{-6}$.

Let W_1 = weight of copper in the coil (disregarding the weight of the insulating material, this is the weight of coil),

W_2 = weight of volume of the coil of copper,

$$\frac{W_2}{W_1} = \frac{a}{\text{section of copper}} = \frac{\rho}{\rho_0}.$$

It is evident that if for any coil whatever we calculate $\frac{L}{R}$ on the assumption that there is no space occupied by insulation, we have only to multiply this by $\frac{W_1}{W_2}$ to obtain $\frac{L}{R}$, taking insulation into account. Another way of putting our result is

$$\frac{L}{R} = \frac{V' \div 1000}{\cdot 728a + 1\cdot 33c + 1\cdot 5b}, \quad . \quad . \qquad (7)$$

where V′ is the volume, in centimetres, of *copper* in the coil. For a coil of given volume wound with given wire, to diminish the ratio of $\frac{L}{R}$, increasing c is nearly of the same importance as increasing b; whereas these are about twice as important as increasing a. In fact $\frac{L}{R}$ is nearly proportional to $\frac{abc}{a+2(b+c)}$; and of course it is less as the wire is finer, because the volume of the insulation is relatively greater.

It may be noted that the empirical formula given in this paper, although seemingly truer when $\frac{b}{a}$ and $\frac{c}{a}$ are smaller, is not true when $\frac{b}{a}$ and $\frac{c}{a}$ are zero. The inductance of one spire of wire whose section is infinitely small is infinite; and it would have been possible to get an empirical formula, fairly correct for ordinary values of a, b, and c, which would give an infinite value for L when $\frac{b}{a}$ and $\frac{c}{a}$ are zero, based on the expression given by Maxwell,

$$\frac{L}{L\pi n^2} = a\left(\log \frac{8a}{R} - 2\right),$$

where R is a function of b and c. Such a formula would, however, be much less useful for ordinary purposes than the one I have given.

III. *On the Measurement of Electromagnetic Radiation. By* C. V. Boys, *A.R.S.M., F.R.S., Assistant Professor of Physics at the Royal College of Science, London,* A. E. Briscoe, *A.R.C.S., and* W. Watson, *A.R.C.S., B.Sc.*[*]

[Plate II.]

At the time that Mr. Gregory described and exhibited his "Electric Radiation Meter"[†], which indicated and measured electromagnetic radiation by the increase in length of a fine platinum wire stretched in a glass tube in consequence of the heat developed by the induced currents within it, one of us expressed the view that it would be interesting to confirm by some independent method the conclusion that the wire was lengthened by heat, as it was not absolutely certain, though exceedingly probable, that the increase in length might not be apparent only and not real.

The following is an account of experiments made by two independent methods, with the view of finding whether the expansion observed by Mr. Gregory was such as might be expected from the induced currents.

First Method.

Imagine two straight wires parallel and near together, but not touching one another or any conductor. Let them be placed parallel to and at a convenient distance from a cylindrical rod of the same length as either of the wires. Then, if electrical oscillations are started in this rod by any well-known method, the two wires will act as resonators, and induced oscillations will occur in them, not necessarily in absolutely the opposite phase to the oscillations in the primary, but at any rate in the same phase in the two wires, so that the electricity in each will surge up and down at the same time. If, then, the ordinary theories of electrodynamic and electrostatic action apply in the present case, which may be true or may not, then the currents going up together and down together will attract one another, while the charges resting chiefly at the two ends, being at one moment + above and − below, and at the next moment reversed, will repel one another. Owing to the harmonic

---

[*] Read June 20, 1890.

[†] Proc. Phys. Soc. vol. x. p. 290.

Figs. 5 to 10.
Scale $\frac{1}{3}$.

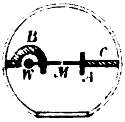


Fig. 9.

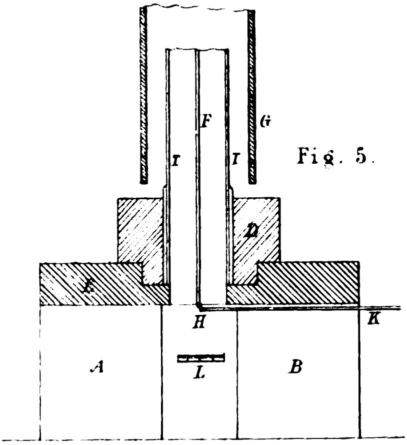


Fig. 5.

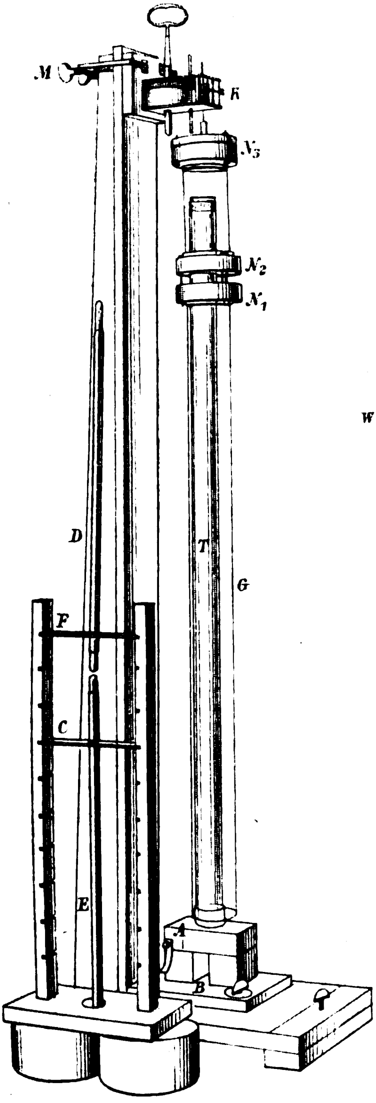


Fig. 11.

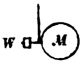


Fig. 10.

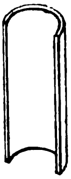


Fig. 7.

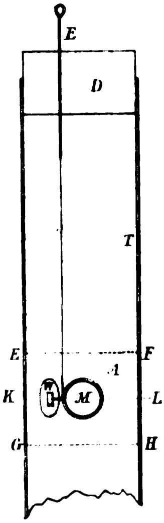


Fig. 8.

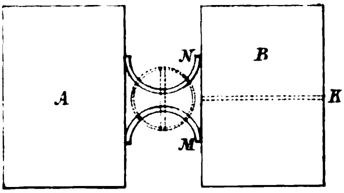


Fig. 6.

Mintern Bros. lith

distributions of the current and the charges, the electrodynamic attraction will be mainly in the middle parts, while the electrostatic repulsion will be mainly at the two ends. Further, both these distributions vary harmonically with the time, and are a quarter-period removed from one another; but the integral effect of each is equivalent to a steady force, and the quarter-period difference of phase has no influence so far as forces are concerned.

In order to realize what these forces are, and what are the relations between them, it will be necessary to consider them separately.

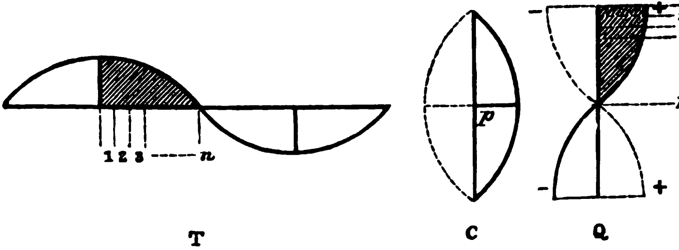
(1) *Electrodynamic Attraction.*—Let the wires be supposed to be very close together, so that the force felt by any element is almost entirely due to those elements of the other wire which are so close to it that the variation of the current strength in those elements along the wire is inconsiderable. Now, if the distance apart is d, if c is the strength of the current at any moment (measured electrodynamically), and if l is the length of the element, the force exerted by the other wire on this element will be $2\frac{c^2l}{d}$ dynes. But as the current at any point varies harmonically with the time, the average force will be necessarily half the maximum force; therefore the average force on the centre elements will be $\frac{c^2l}{d}$, if c is the maximum current at the centre.

Since, also, the current varies harmonically along the wire, the average force per unit length over the whole wire will be half that at the centre; so that the attraction between the wires will be $\frac{c^2\mathrm{L}}{2d}$, where L is the whole length of the wire.

(2) *Electrostatic Repulsion.*—In the same way, if q is the quantity of electricity (measured electrostatically) on unit length at any part of the wire, the force of repulsion upon an element at any moment will be $2\frac{q^2l}{d}$, the average force on this element will be $\frac{q^2l}{d}$, and the total repulsion between the wires will be $\frac{q^2\mathrm{L}}{2d}$.

Now to find the relation between these it is necessary to remember that the wires are resonators; that is, the oscillations follow the natural period and are not forced. Also that the total quantity of electricity on one half of the wire is the integral of the current that has passed the middle point. Although there is no difficulty in finding the relation between the attraction and repulsion, the following graphic method may be worth giving. In order to simplify matters, the resonator will at first be supposed of such a length that the oscillations occur at the rate of one a second, *i.e.* it will be supposed to be $\frac{1}{2}$ of 3×10^{10} centim. long; also the maximum current at the centre will be supposed unity. Now let the vertical line in C (fig. 1) represent the wire, and the distance of the harmonic curve from it at any point the strength of the current at that point when it is at a maximum.

Fig. 1.



In the same way, let the distance of the harmonic curve in Q from the wire at any point represent the charge at that point a quarter-period later, at which time it is at a maximum; then the total quantity on one end, represented by the shaded area, is the integral of the current which has passed the middle point *p*. Now this current varies harmonically with the time, as shown in the time-diagram T. This is divided into four equal parts. Thus the first represents (say) an upward current rising from zero to its maximum; this just neutralizes the previously charged wire. Then, in the second period, the continuation of the current produces an equal and opposite charge, which is a maximum when the current has become zero. In the third period, the current in the opposite direction has again neutralized the charges by the time that it is a maximum; and in the fourth period, it charges the upper end negatively and has ceased to flow when the negative charge is a maximum.

Now divide the second period of the time-diagram of current, T, into n ($=\frac{1}{4}$ of 3×10^{10}) elements, and the upper half of the wire in the quantity-diagram, Q, into the same number of equal parts, which will therefore each be one centimetre long. Now, since the total quantity of electricity represented by the shaded part in Q is equal to the time-integral of the current which is represented by the shaded part in diagram T; and since also each of these is bounded by a quarter-period of a sine curve, and is divided into an equal number of elements, it is clear that each space-element in Q, $1, 2, 3, \ldots n$, must equal the corresponding time-element in T ; or, in words, though the quantity of electricity that has passed the point p in the first element of time is spread over the upper half of the wire in Q, and so on, yet at the nth element we may consider that that which passed during the first element of time is stored on the element 1, that which passed during the second element of time is stored on the element 2, and so on. Now, since the current-strength in the first element of time is 1, and since this element of time has been made equal to $1/(\frac{1}{3} \times 10^{10})$ second, the quantity of electricity on the end centimetre of wire is $1/(\frac{1}{3} \times 10^{10})$ electrodynamic unit, or 1 electrostatic unit. This will be repelled by the wire opposite with the same force as that with which the current in the centre element of one wire will be attracted by that in the other wire, both when each is at a maximum and through all the harmonic variations. In the same way, the next element near the end in Q will correspond to the next element near the middle in C, and so on over the whole length of each ; and so, since the electrodynamic attraction over every element in time and space is equal to the electrostatic repulsion in the corresponding element, removed both in time and space by a quarter period, the total force due to one is equal to the total force due to the other. If any other length of wire is taken, and of course its corresponding period, since the number or length of the elements in both T and Q must vary together, the same result will follow; and if any other strength of maximum current is taken, the attractions and repulsions will be changed to an equal extent, and therefore, if two straight wires are placed so close together that the distance between them is very small compared to their length, they will, when their natural electrical oscillations are induced in them, on the whole neither attract nor repel one another.

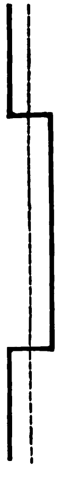
It follows from this that, if the oscillations be forced and made too slow, the accumulation of charge on the ends will be greater for a given maximum current than it would be if the period were natural, and therefore the repulsion should overpower the attraction. In the same way, with too rapid vibrations the attraction should overbalance the repulsion. Also since, if the wires are not close together, the electrostatic repulsion will have fallen away in a higher ratio than the electrodynamic attraction, it would appear that resonators which balance when close should attract one another when at a distance, or that resonators subject to forced oscillations which repel when near might attract when at a distance.

Prof. Fitzgerald, when the above theory of the mutual action of resonators was put to him, at once suggested that if the molecules of a body were subject to electrical oscillations it would be possible to imagine a physical cause for their supposed attraction when far, and repulsion when close.

In order to test the matter experimentally, the wires were made of a cranked form, as shown in fig. 2, so that the ends were on one side of a vertical axis (dotted), and the middle part on the other side. Then, if one is held close to the other, and if the second one is free to turn about the vertical axis, the attraction of the centre and the repulsion of the ends should conspire to produce rotation in the same direction[*].

The first experiment was made with a pair of copper wires each one foot long, bent so that the end parts were each 2½ inches, the middle part 5 inches, and the horizontal part 1 inch. This was suspended in a glass jar by a quartz fibre, so that the vertical axis of

Fig. 2.
¼ full size.



---

[*] It may be worth mentioning that the invention of this method was made in a particularly vivid dream, in which I found myself at the blackboard in the physical lecture-room, explaining that if the two wires were

rotation was halfway between the ends and the middle portion. An ordinary galvanometer-mirror was attached to the rigid glass hook from which the wire was suspended, and this could be seen through a plate-glass window in a torsion-box above the jar. The oscillator consisted of two pieces of brass tube half an inch in diameter with hemispherical ends, each 6 inches long in all. This was placed in a vertical position two or three feet away from the wires, and in the plane of the suspended wire, so that it should not act directly upon it. The two parts of the oscillator were connected by very fine wires with the terminals of a Voss machine. Then it was found that when the halves of the oscillator were separated too far for a spark to pass, there was no deflexion, while when they were gradually made to approach one another, no motion was visible on the scale until sparks passed, and then the motion was evident enough, the light even going in favourable cases off the scale in the right direction.

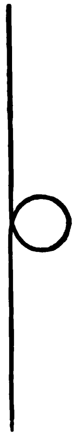
However, this was only a preliminary experiment made with materials at hand. It was clear that the material which was most suitable was that which had the highest conductivity for its density, namely aluminium, and that only the outer surface of the wire could be efficacious, since the oscillations were at the rate of about 500,000,000 a second. Accordingly, thick aluminium wire was rolled into thin ribbon, which was then drawn through successive holes in a draw-plate until a perfect tube, cracked down one side of course, was formed.

We used resonators made of this material in the next experiments, which were made with the view of seeing

---

close together, the currents in each would be going up and down together, and so would attract one another, and that in order to make this attraction evident, the wire should be suspended near the axis of rotation, for the reason that where quartz-fibres can be employed there is a greater gain owing to the reduction of moment of inertia than loss by reduction of statical moment. I did not in the dream see that the charges should repel, or the necessity for the cranked form of wire. I may say that this was after Mr. Gregory had described his instrument, but the idea of looking for mutual attraction between a pair of resonators had not before occurred to me.—C. V. B.

Fig. 3.



whether the couple observed was due equally to the repulsion of the ends and attraction of the centre, or if one preponderated over the other. The fixed cranked resonator was then replaced by a straight one, which was held at first close and opposite to the ends of the suspended cranked resonator, when a strong repulsion was observed, as expected. It was then held in the corresponding position opposite the centre part, when, instead of the expected attraction, feeble repulsion was observed; and finally, on being placed equidistant from the two, the ends were repelled away.

Since this appeared to show that the electrostatic repulsion was much greater than the electrodynamic attraction, we tried the following experiment, in which the electrostatic action was reduced to practically nothing. A pair of resonators were made of the form shown in fig. 3, and one was weighted with a non-conductor and suspended so that the straight part was in the axis of rotation. The other was then held with the curl close and parallel to the curl of the other, and so that at first the current in each should go round the same way, and then in opposite ways. No action was observed in either case. When also cranked or straight resonators were entirely immersed in water, no effect at all was observed.

We have given a short account of the early experiments because at first they were so completely in accord with the result of the theory put forward, though the action was more powerful than had been expected, that for a time we were deceived as to the real cause, and we wish to guard others from falling into what is after all a very obvious error. The later experiments showed either that the theory was only true as regards electrostatical repulsion, the electrodynamic attraction of the momentary currents being *nil*, or else that the results already obtained were spurious.

We have stated that the two resonators were suspended in a glass jar, outside which the primary oscillator was placed, and that while no repulsion was observed during the gradual charging of the oscillator-ends to a higher potential than that employed with the sparks, yet the moment the sparks passed when the ends were nearer together, the suspended resonator was observed to move in the expected direction. Now if the glass jar were a conductor, the oscillator-ends might be charged to any extent and no action would be felt inside ; if, on the other hand, it were a perfect insulator, then during the charging of the oscillator-ends before the spark the resonators would have induced charges in their ends which would make them repel one another. The glass jar exposed to the moisture of the air was a sufficiently good conductor, on its outer surface at any rate, to be able to screen the statical action of the charges on the oscillator-ends which were by the arrangement only slowly made to vary, and so the repulsion which might have been looked for was not observed ; but the moment a spark passed in the primary, the conditions were changed. The fall of potential was instantaneous, but the screening charge on the glass surface, which the previous moment produced within the jar an effect equal and opposite to that of the charges on the oscillator-ends, was unable instantly to disappear owing to the imperfectly conducting surface of the damp glass, and thus at this moment the resonators were in the condition that they would have been in had there been suddenly developed on the oscillator-ends equal and opposite charges, but with no glass jar between. This then was the cause of the repulsive impulse which was observed at each spark when they followed one another at intervals. When, however, a torrent of sparks was produced, the glass surface would take up a screening charge equivalent to the mean charge on the oscillator-ends ; then, when the oscillator-ends had charges either above or below the mean charge, the statical action on the resonators would be the same as that due to the excess or defect on the oscillator, so that the steady repulsion observed was due to induction alone, and had nothing whatever to do with electrical oscillations proper.

In order to perfectly screen off this induction effect, and yet leave the action of the true electrical oscillations, it was necessary to so increase the conductivity of the surface of the glass jar that it would have time to acquire a screening charge which should follow variations of potential accompanying the sparks at the rate of, say, 10 or 20 a second, and yet leave it so imperfect as to be practically an insulator to oscillations at the rate of 500,000,000 a second, which corresponded to the apparatus in use.

Prof. J. J. Thomson has shown* that the conductivity of liquids may be determined by finding what thickness of a sheet of liquid is just sufficient to prevent oscillations of a known rate from exciting a resonator on the other side, and that the thickness must be proportional to the specific resistance. For instance, he found that a layer of dilute sulphuric acid three or four millimetres thick was just opaque to oscillations at the rate of 100,000,000 a second. It would therefore require a greater thickness to be opaque to oscillations of 500,000,000 a second, and a very much greater thickness for water to be opaque to oscillations of this rate. A duster, therefore, soaked in water, though it might reflect some fraction of this radiation, should be practically transparent to it. Nevertheless a wet duster would be amply sufficient to screen the action of true harmonic variation of potential of the frequency of the sparks, *i.e.* 10 or 20 a second. As a matter of fact, these variations of potential are not simply harmonic, but since by Fourier's principle the non-harmonic periodic change is equivalent to a series of true harmonic changes at successively higher rates all superposed, the wet duster should completely screen all the lower of these components up to a certain point, and probably far enough. On wrapping up the jar with a wet duster all effect was found to cease, and the spurious character of the deflexions was demonstrated. It is possible that these are not the only experiments with electrical oscillations in which a wet duster or its equivalent might be found of service.

The apparatus was then remade with the view of obtaining very great delicacy. To make the resonators aluminium wire was rolled into ribbon of ·2 × ·001 centim., so that each centimetre in length weighed only half a milligram. Two pieces 30 centim. long were fastened in the cranked form previously described on very light frames made by cementing to almost capillary thin glass tubes cross bars cut from microscope cover-glass, so that the ends of these cross bars held the corners of the cranked pieces of aluminium ribbon. One of these was suspended in the same apparatus by a fine quartz fibre, and all was arranged with the view of obtaining a delicacy comparable with that employed by one of us in the Cavendish experiment. It was now impossible to keep the spot of light at rest, owing mainly to air-currents in the jar set up by differences in temperature of the sides of the jar which could not be avoided in the schools, and also to the perpetual tremor. In the winter, however, the elementary laboratory on the other side of the Exhibition Road was not being used, and so everything was transferred to a small room there, where a fair degree of quiet and uniformity of temperature could be obtained. The Voss machine was replaced by a large induction-coil, which gave a greater torrent of sparks. Then, to get more energy in each spark, the primary oscillator, made of 12 millim. tube, was replaced by a pair of spheres 100 millim. in diameter, with small spheres on short sliding rods, between which the sparks passed. The scale was placed at a distance of 281 centim. from the mirror. Then, when the wet duster was placed round the jar and the sparks made to pass, deflexions of 5 to 10 millim. were observed. Fearing, however, that even the wet duster failed to completely screen the induction-effects due to the rise and fall of potential of the primary or rather some of the higher harmonics already alluded to, we soaked it in dilute sulphuric acid instead of water, so that no doubt should exist as to its sufficient conductivity to quench the sham effect while it still remained unable to stop more than a small proportion of the waves of 500,000,000 a second. With this precaution all definite movement of the spot of light ceased. It was seldom that air-currents did not cause a gentle hovering of the index a few millimetres either way, which, however, had no relation to the starting or stopping of the sparks; on one or two frosty days, however, when the temperature was very uniform, there being no fire, the spot of light was so steady that a movement of even $\frac{1}{10}$ millim. on the scale would have been observed, but we have based our calculation on the certainty that the deflexion was not $\frac{1}{5}$ millim.

---

* Proc. Roy. Soc. vol. xlv. p. 269.

In order to make sure that this was a true negative result the duster was removed, when at once the index was driven violently off the scale, and then a Hertz resonator tuned to the oscillations of the primary was placed in the jar and surrounded by the same acid duster when the sparks in the gap were evident, showing that the proper electromagnetic waves did penetrate into the jar as supposed.

The turning moment corresponding to a deflexion of $\frac{1}{5}$ millim. may be found from the following particulars :—

Period of oscillation of mirror and stem (oscillator removed), ·947 second.

Period of oscillation with wire ring hung on, 4·13 second.

(Wire ring 1·6 centim. diameter outside, made of wire ·122 centim. in diameter; mass ·4635 gramme.) Hence

$$\text{Couple} = 1{\cdot}1 \times 10^{-5} \text{ C.G.S. units, and}$$

$$\text{Average value of couple} = 3{\cdot}7 \times 10^{-7} \text{ C.G.S. units per centim. of wire of resonator,}$$

which corresponds to a force of about 3×10^{-7} dynes, since the distance from the axis is $1\frac{1}{4}$ centim. This is less than a two-hundred-millionth of the weight of a grain, or a three-thousand-millionth of the weight of a gramme.

Since the wires were only ·2 centimetre apart, this shows that, if the currents in the wires, instead of being oscillating, had been simply ordinary steady currents, those currents could not have reached the value of ·00017 C.G.S. electromagnetic unit or ·0017 ampere.

Now the moment, if the theory advanced is correct, is due partly to electrostatic repulsion and to electrodynamic attraction which are equal to one another, but owing to the harmonic distribution of both, both in time and in position, and owing to the fact that the weaker part of each of these forces is tending to counteract the stronger part, the total average resultant force is by no means double or even equal to that due to a steady current alone of a strength equal to the actual maximum.

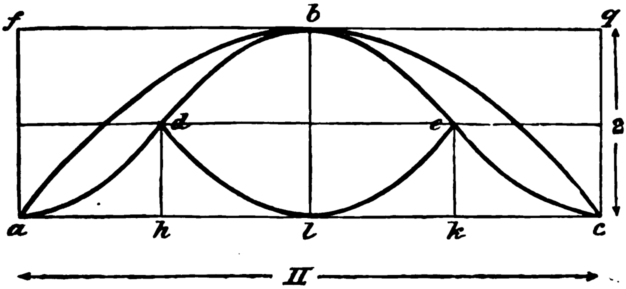
To find the relation it is necessary to first consider the effect of the harmonic distribution of either, say the current, along the wire. The square of this at any point is a measure of the force at that point, so that the force varies from zero at the ends to its full value in the middle, having passed its half value at the points where the wire crosses the axis of rotation.

Thus, in fig. 4, the ordinates of the upper curve *a b c*, which is half a sine curve, represent current strength, those of the lower curve *a d b e c*, which is a complete sine curve on half the scale, but shifted vertically by the amount of its own maximum ordinate, since

$$\sin^2 a = \frac{1-\cos 2a}{2},$$

represent the force due to the current at each point. The rectangular area *f g c a*, equal to 2π, measures the total force due to a uniform current of the maximum value. The areas *a d h* and *c k e* are the forces due to the ends of the wires in the wrong direction, and these together

Fig. 4.



$$= \frac{\pi}{2} - \int_0^{\pi/2} \sin\theta\, d\theta = \frac{\pi}{2} - 1.$$

The area *h d b e k* is the force due to the middle half of the wire in the right direction, and this

$$= \frac{\pi}{2} + \int_0^{\pi/2} \sin\theta\, d\theta = \frac{\pi}{2} + 1.$$

The difference of these, which is the area *d b e l*, is the balance in the right direction, and this $=2$. But the force due to a current of the maximum strength over the whole length of the wire is 2π, so the force in the cranked conductor due to the harmonically distributed current and the opposite effect of the ends is $\frac{1}{\pi}\times$ that due to uniform maximum current over the whole wire. The current is not steady, but varies harmonically with the time, so on this account the value must be halved again, but this halving is exactly balanced by the doubling which results from the fact that the electrostatic exactly equals the electrodynamic effect.

We have seen that the force was not so great as that due to a steady uniform current of ·0017 ampere; therefore the maximum current in the centre could not, if the oscillations continued uninterruptedly, have reached the value of $\sqrt{\pi} \times \cdot 0017$ or ·003 ampere. Since in the most violent torrent of sparks that can be produced the time during which the oscillations last is very small compared to the interval between two sparks, it is quite possible, and even probable, that the actual current during the earlier part of each spark greatly exceeded this, but this is a superior limit to the value of the square root of the mean square of the strength of the current in the particular case.

It should be noted that two errors, each of which is numerically insignificant, have, for convenience, been made. One is the supposition that the forces on the elements near the ends of the resonators are the same as though they were continued together with the stationary wave. As a matter of fact, it is only the last few millimetres for which there is a falling off. The second error is made by supposing that the middle half of the wire is all at one distance from the axis on one side, and the two end quarters at an equal distance from the axis on the other side. As a matter of fact, the wire has to cross over, but as the two forces almost exactly balance on the small part which crosses the axis, it was not worth while to complicate the simple statement of conditions illustrated by fig. 4 for the sake of an imaginary degree of accuracy which the nature of the observations would not justify.

Second Method.

As the first method, carried out with all the delicacy which under the conditions we could make use of, gave no promise, we determined to find by some direct process whether the heat which Mr. Gregory had inferred from the expansion of the wire was real. For this purpose we made use of one of the most delicate heat-measuring instruments that exist, which is specially suitable for detecting very small quantities of heat diffused over a great length of wire, namely the convection air-thermometer of the late Dr. Joule*. With this instrument he was able to detect the direct heating-effect of the moon's rays unconcentrated by lens or mirror.

His instrument consisted of a tube about 2 feet long and 4 inches in diameter, divided longitudinally by a blackened pasteboard diaphragm, leaving spaces at the top and bottom. In the top space a piece of magnetized sewing-needle, furnished with a glass index, was suspended by a single silk fibre. Any excess of temperature on one side of the partition caused an air-current, which at the top passed from the hot to the cold side and deflected the needle.

We have made use of Joule's instrument, not exactly in the original form, but with two classes of index, and with an essential improvement which makes it extraordinarily sensitive without greatly affecting its simplicity. The first class of index depends for its working on the fact that when a small piece of phosphorus is placed in air it gradually oxidizes, and the oxide formed slowly falls, if the air is still, in a well-defined column, appearing like smoke. If there is any very small motion of the surrounding air, this column of phosphorous oxide will be deflected. It thus forms an excessively delicate and light index for detecting movements in the air.

To overcome the difficulty of keeping the index at rest, owing to radiation from neighbouring bodies or to draughts of varying temperature, we have surrounded the divided tube by another, which was kept rotating continuously by clockwork. If any part of this guard-tube becomes slightly heated, it is carried round and made to heat the instrument equally all round, and so it produces no effect.

As the determination on theoretical grounds of the most suitable dimensions presents considerable difficulties, we decided to make a series of tubes of different sizes and compare their sensibility. The object of the instrument being to measure the heat developed by the electromagnetic radiation in a wire within the instrument, it was necessary to make both the tube and guard-tube of some insulating material, such as glass.

---

* Proc. Manchester Phil. Soc. vol. iii. p. 73.

The tube to be used was cleaned, and a roll of brown paper about 1 inch long was pasted round one end. This fitted tightly into a mahogany ring, D (Plate II. fig. 5). The tube was divided longitudinally by a partition of cardboard (F), which was made to fit tightly into the tube. This partition extended from the bottom to within about an inch of the top, which end was closed by a cork.

The projecting end of the mahogany ring (D) fitted into a corresponding hole in a piece of wood (E), which was itself supported on two blocks (A and B). The tube (represented by the dotted circles in the plan, fig. 6) came directly over this space between A and B, the partition being parallel to their adjacent faces.

Between A B, and immediately beneath the tube, two glass semi-cylinders were placed, with their convex surfaces facing each other. They are shown in plan in fig. 6 (M N), and one is shown in perspective in fig. 7. At 2 centim. from the top of one of these semi-cylinders a small paper scale (fig. 5), divided in half-millimetres, was attached. When a current of air passes from one side of the tube to the other, it has to pass through the narrow space between these cylinders.

A hole was bored through the top of B, through which a capillary glass tube could be passed. This tube was filled with phosphorus, and the glass broken away for about one millimetre, so as to expose a clean piece of phosphorus whenever the stream of oxide had to be observed. The tube was then passed through the hole (K), so that the exposed piece of phosphorus came just beneath the partition and between the glass cylinders. It is shown in place at H in fig. 5.

By this means the stream of phosphorous oxide was formed at the narrowest part of the air channel, and so was affected to the maximum extent by any passage of air from one side of the partition to the other. The resulting deflexion of the phosphorous-oxide stream could be read with an accuracy of about one-tenth millimetre. In this form, when the phosphorus index is deflected it is carried to a part where the channel is much wider, owing to the curvature of the glass half-cylinders.

On this account the deflexion of the index is not any simple function of the rate at which the air is passing. With a view to do away with this want of proportionality, and also, if possible, to render the instrument more sensitive, the glass half-cylinders were replaced by a pair of narrow flat glass plates, near together, with a wide channel on each side communicating with the tube above. The instrument was, however, found to be less sensitive in this form than when the half-cylinders were used.

The sensitiveness of the instrument could, to a certain extent, be altered by moving M and N (fig. 6) further apart, so that the phosphorus stream fell in a wider channel.

The divided tube was surrounded by another glass tube (G, fig. 5), which was about 10 centimetres longer. This formed the guard-tube, and was suspended from one of the axles of a clock (K, fig. 11), which made one revolution in about eight seconds.

By means of three levelling-screws the guard-tube was made to hang free of the inside tube, so that it did not touch it at any part of its revolution. Thus no heat due to friction was developed near the divided tube.

The phosphorus could not be placed at the top of the tube, as the falling column of oxide was found to go on one side of the partition, and its weight was sufficient to keep up a down-current of air on this side and an up-current on the other side, and so make the instrument unstable.

Inside the divided tube, and on one side of the partition, two fine platinum wires were stretched, reaching from the top to the bottom. They were connected together at the top, and at the bottom were brought out and connected to two binding-screws fixed on the ring D (fig. 5). By means of this wire, whose resistance was measured, a known amount of heat could be developed in the tube by passing a current of known intensity. The current was, in all the experiments, supplied by a single Daniell cell, which contained in its circuit, beside the wire of the instrument, a resistance-box and a key.

The clock was kept working for about half an hour before the commencement of an experiment, in order to equalize the temperature within the tube; and then a fresh piece of phosphorus was introduced. When the phosphorus index was quite steady, and approximately at the centre of the scale, the reading was taken (zero-reading), then resistance was unplugged in the box, and the current passed till the index had again come to rest. The current was then stopped, and another zero-reading taken.

In testing the different tubes the largest resistance which would give a readable deflexion was in each case ascertained. The resistance of the cell and the connecting-wires could be neglected in comparison with the external resistance.

In the different tubes the total heat developed was not the same when the same current was passed, as the wires were of different lengths. This, however, was not taken into account in comparing the tubes together, as for the measurement of the electromagnetic radiation it was necessary to get the greatest possible sensitiveness per centim. of the wire.

It is hardly necessary to give particulars of the different tubes experimented upon. It is sufficient to say that, while the length of the tube was of little consequence, it was found that a wide tube was much more sensitive than a narrow one. The results obtained in a particular case are given.

Tube No. 3 :—Length 82, diameter 5 centimetres. Resistance of wire 29 ohms ; outside resistance 500 ohms. Deflexion 1 millimetre.

To calculate the amount of heat corresponding to this deflexion, we have, taking the E.M.F. of a Daniell cell as 1 volt,

$$\text{Heat developed per sec.} = \frac{RC^2}{4{\cdot}2}$$

$$= \frac{29}{529^2} \times \frac{1}{4{\cdot}2} = {\cdot}000{,}024 \text{ calorie}$$

over the whole length of the tube, or ·000,000,3 calorie per centimetre of the tube.

While this very small degree of heating was within the range of experimental certainty so long as the clockwork was going, it was impossible to keep any index visible at all when the clock had stopped, even though several concentric screens were placed round the tube.

The apparatus in this form was subject to the great fault that the phosphorus refused to form the oxide at a temperature below 15° C., which prevented the instrument being used on many days.

With the view of getting over this difficulty with the phosphorus index, and if possible of obtaining greater sensitiveness, a form of index was adopted which was a modification of Dr. Joule's arrangement.

The tube was closed at each end, the partition reaching right up to the cork at the top and leaving a space of about fifteen centimetres at the bottom. Twelve centimetres below the top of the partition a hole was punched in the card. Into this hole a short cylinder of paper was glued. This is shown at A in figs. 8 & 9. A plane mirror, M, fitted into this tube, allowing about 1 millimetre clearance all round. It was fastened to a cross bar, at the other end of which a counterpoise (W) was fixed.

The mirror and counterpoise were suspended by a fine quartz fibre from a pin (E, fig. 8), which passed through the cork used to close the tube. When the mirror is at rest it almost completely fills up the hole in the partition, and forms a very delicate means of detecting any differences of pressure in the air at the two sides of the partition.

A small recess (B, fig. 9) was made in the partition for the counterpoise, and covered in so that it was not exposed to the current of air. The arrangement is shown in elevation in fig. 8, and in section through K L in fig. 9. Fig. 10 shows the mirror and counterpoise separately. A hole was ground in the tube immediately in front of the mirror, into which a lens of a metre focal length was cemented.

In this form of the instrument the guard-tube, if continuous, would have prevented the formation of a clear image. It was therefore necessary to divide it into two portions. Rings of mahogany (N, fig. 11) were cemented to the top of the lower portion and to the top and bottom of the upper. The two rings N_1 N_2 were connected together by three bolts, of such a length as to leave a space of about two centimetres between the rings. This break in the guard-tube came just on a level with the mirror, and thus allowed the beam of light to pass freely. The passage of the bolts before the mirror produced no inconvenient effect on the image formed on the scale. The ends of the wire were brought out at the bottom of the tube, leaving about half an inch free. When using a continuous current, two brass clips (A, fig. 11) were placed on these ends so as to connect them to a pair of binding-screws. When using the oscillator these clips were turned down (as in the figure), so that the wire should not be connected with any body having capacity.

The following are the dimensions of this tube (No. 4):—

Length 126 centimetres, diameter 4·7 centimetres.

Length of wire 2 × 103 centimetres.

Resistance of wire 38 ohms.

Diameter of guard-tube 7 centimetres.

Diameter of mirror 1·2 centimetre.

Length of fibre 9 centimetres.

As shaking interfered with the working of the instrument, the experiments were performed in a vault, which, however, was by no means steady.

The following are the results obtained when testing the sensibility of the instrument (scale at 100 centimetres from mirror):—

Added Resistance.	Total Resistance.	Deflexion.
ohms.	ohms.	centimetres.
500	538	3·3
762	800	1·6

In the case of 800 ohms total resistance we have,

$$\text{Heat developed in wire per sec.} = \frac{1}{\overline{800}|^2} \times \frac{38}{4{\cdot}2}$$
$$= {\cdot}000{,}014 \text{ calorie.}$$

This is equivalent to one calorie developed in 19 hrs. 42 min. Or we may say that one calorie per day developed by the whole wire could certainly be detected; that is, one calorie in 103 days per centimetre of tube.

The oscillator used to set up the electromagnetic radiation consisted of two hollow brass cylinders (D E, fig. 11), fixed to ebonite cross pieces (F, C), which could be placed at different heights in a wooden stand. Short brass cylinders, with well-rounded ends, fitted into the ends of D and E, and allowed the length and sparking-distance to be altered. These cylinders were connected by very fine wire with the terminals of the large Ruhmkorff's coil belonging to the Physical Laboratory. The length of the oscillator was the same as that of either of the wires in the tube, and it was placed so as to be in the plane of the partition of the tube, at a distance of about 30 centimetres (fig. 11). The best length of spark was about half a centimetre.

At first, on separating the knobs of the oscillators, so that no spark passed, the instrument showed a large deflexion in the direction of cold. This was due to electrostatic attraction between the oscillator and the mirror. To prevent this action a piece of tin-foil (E F G H, fig. 8) was pasted round the tube at the level of the mirror, a hole being cut in it the size of the lens, which effectually put a stop to this action.

The instrument now indicated a considerable amount of heat developed in the wire whenever the coil was worked.

The effect of altering the capacity of the oscillator by adding pieces of tin-foil to the ends was tried, but with no beneficial result.

A wet duster was hung between the oscillator and the instrument for the purpose of seeing whether the radiation would pass through. The deflexion produced was only slightly less than when no duster was interposed.

In order to measure the heat developed by the induced currents in the wire, the deflexion produced when the coil was worked was noted; then a steady current was sent through the wire, and the resistance in the circuit altered till an equal deflexion was obtained. The readings obtained with the oscillator were somewhat varied, owing to small changes in the character of the sparks. Another difficulty was caused by the coil often refusing to work continuously for the time necessary to get a reading (about 1½ minute), as the contact-breaker became burnt by the sparks.

The following are the deflexions obtained in centimetres :—

45 ; 44 ; 37 ; 36 ; 35·6 ; 36·9 ; 44·8 ;
45·5 ; 40·2 ; 40·2 ; 39.

Mean 40·4 *centim.*

One Daniell cell working through a total external resistance of 115 ohms gave the following deflexions :—

41·0 ; 40·0 ; 39·3 ; 40·0 ; 41·0.

Mean 40·3 *centim.*

We may therefore say that the heating-effect due to the oscillations (oscillator at 30 centim.) is equal to that produced by one Daniell cell when the external resistance is 115 ohms, and the working resistance 38 ohms.

Therefore the heat produced per sec.

$$= \frac{38}{115^2} \times \frac{1}{4{\cdot}2} \text{ calories,}$$

$$= {\cdot}000685 \text{ calorie;}$$

or it would take 24 minutes 20 sec. to produce one calorie.

The steady E.M.F. between the terminals of the wire which produces the same effect as the oscillations is ·33 volt. Mr. Gregory gives ·5 volt as the equivalent E.M.F. in his experiments. Though the conditions were not sufficiently similar to make the numerical results strictly comparable, nevertheless the fact that the figures obtained are of the same order goes to show that the expansion observed by Mr. Gregory was a true expansion due to heat, and that his results were real.

In calculating from the heat developed the mean square of the current produced in the wire by the electrical oscillations, it is necessary to take into account the fact that these rapidly alternating currents are not uniformly diffused throughout the cross section of the wire, but only pass through a thin stratum at the surface. On this account the resistance which the wire offers to these rapidly alternating currents is greater than that which it offers to steady currents.

Lord Rayleigh * gives a formula for calculating the effective resistance of a wire of round section to an alternating current of given frequency. It is

$$\mathrm{R}' = \sqrt{(\tfrac{1}{2} p l \mu \mathrm{R})},$$

---

* Phil. Mag., May 1886, p. 388.

where R′ is the effective resistance; R the resistance to steady currents in C.G.S. units; p 2π times the number of oscillations in one second; l the length of the wire; and μ the magnetic permeability of the substance of which the wire is composed.

In the following calculation one of the wires only is considered.

The length of the primary oscillator is 109 centimetres, and therefore the wave-length of the radiation given out is 218 centimetres. If n=number of oscillations per sec., then

$$n=\frac{v}{\lambda}=1{\cdot}376\times 10^8.$$

The length of the wire (l) is 103 centimetres, and its resistance 19 ohms$=19\times 10^9$ C.G.S. units.

In the case of platinum μ may be taken as unity.

From this it follows that

$$R'=29 \text{ ohms},$$

and since H (half the heat developed in the two wires) $={\cdot}000342$ calorie per second

$$C^2=\frac{JH}{R'}$$

$$={\cdot}000049.$$

Owing to the harmonic distribution of the current along the wire, in order to produce a given amount of heat in the whole wire, the strength of the current at the centre must be $\sqrt{2}$ times as great as that which would be necessary if the current were of uniform strength in all parts of the wire.

∴ the square root of the mean square of the current at the centre $={\cdot}01$ ampere.

While it is impossible from the heat found to form any conclusion as to how fast heat was being generated during each spark, or what fraction of the whole time the oscillations lasted, yet here, as in the former case, the final result depends on the mean square of the current. We found by the first method that the square root of the mean square was certainly not ·003 ampere, if the theory was correct; while by the heat we find the square root of the mean square to be ·01 ampere. The arrangements in the two cases were not strictly comparable, for in the first the oscillator was only 30 centimetres long, while in the second it was 109 centimetres, and because the distance of the primary from the secondaries was relatively greater in the first than in the second case. We intended to have made a second set of measures in which the suspended cranked oscillator should itself be in the tube of the convection-thermometer together with the fixed cranked oscillator. In this way the force, if any, between them could be observed and measured at the same time that the deflexion of the mirror due to the heat was read; then, by making contacts with the ends of the fixed cranked oscillator alone, and passing a current through it so as to produce the same deflexion, all the data would be obtained to tell for certain whether the absence of force, or, if force was observed, if the actual force was that which the provisional theory set out at the beginning of this paper indicated. The discrepancy of 3 : 10 in the present instance cannot be taken as proving that the theory is wrong, and that the forces ought not to be observed, but it does clearly show that they must be excessively minute, and that, if they exist, it is only by great refinement that they can be made evident. No doubt, as Prof. Fitzgerald suggested, the use of concentrating cylindrical mirrors of proper form would greatly increase the sensitiveness of the whole apparatus.

We have been prevented from completing this work because the elementary laboratory was required, and it is impossible to carry on any really delicate research requiring uniform conditions and steadiness in so unsuitable a building as the Science Schools. We have therefore published our results in their present imperfect condition, partly because we believe that the methods employed may be useful to others, but chiefly because we hope that some physicist who has the advantage of having a steady underground room may be induced to make the experiments under more perfect conditions, and so carry the matter to a successful issue.

Dr. Lodge asked Sir William Thomson whether, when electric pulses travel along parallel wires with the velocity of light, any action could exist between them, for two charged spheres travelling together at that velocity exert no mutual attraction or repulsion.

In reply, Sir William said he was inclined to think Mr. Boys' treatment of the subject was in the main correct, but it was quite possible that at such velocities the ordinary laws might be modified by the fact that the time taken for the force to be propagated from wire to wire is comparable with that required for the pulse to travel the whole length of the wire. As an example of the peculiar effects of rapid discharges he said he had seen two copper wires which had been flattened against each other by lightning.

Mr. Boys thought that in his resonators a condition analogous to stationary waves would exist, for the pulses are reflected from the ends.

Dr. Lodge said he had that afternoon observed the action of parallel strips when Leyden-jar discharges were passed through them. The strips gave a kick at each discharge.

Mr. Gregory mentioned that in trying to increase the sensitiveness of his meter so as to measure the variation with distance, he had found that two resonators in proximity interfered with each other. He had, however, succeeded in increasing the sensibility about five-fold.

Prof. Worthington asked if it was possible to measure the energy of the oscillator, and also whether the quantity caught by the resonator could be estimated from the solid angle it subtended at the source of energy, wherever that might be.

Prof. Perry considered it easier to infer the energy of the source from that received by the resonator.

Dr. Lodge said the energy of the source could be easily measured. The power radiated was enormous whilst it lasted, vastly exceeding that of tropical sunshine; if it could be made continuous, the apparatus would soon be red hot. The energy radiated, he said, converges on the resonators, and hence the solid-angle method of estimating the amount received would be erroneous. Moreover, the source was not at the oscillator, but at a quarter wave-length from it, and most of the energy returns to the oscillator; only a small fraction is splashed off and sent into space. Small oscillators radiate powerfully because the quarter wave-lengths are small; whereas the slow oscillators, or alternators used commercially, radiate very little of their energy. The exact law of variation of intensity of radiation with distance was rather complicated, but the theory had been completely worked out by Stokes in 1848.

Mr. Blakesley thought the energy that returns to the oscillator would be available for subsequent radiations.

Dr. Lodge pointed out that wires or other resonators placed within the quarter wave-length would intercept part of the returning energy.

IV. *On Certain Relations existing amongst the Refractive Indices of some of the Chemical Elements. By the* Rev. PELHAM DALE, *M.A.* (Abstract.)

THE paper contained calculations published by the author last year, supplemented with fresh data obtained from Muir's 'Principles of Chemistry.'

V. *Additional Notes on Secondary Batteries. By* Dr. J. H. GLADSTONE, *F.R.S., and* W. HIBBERT, *F.I.C.*

DURING the publication of a series of papers by Dr. Gladstone and Mr. Tribe in 'Nature,' 1882, on the "Chemistry of the Planté and Faure Accumulators," a discussion arose as to whether the sulphate of lead, to which they attributed so important a part in the reaction, could be reduced by electrolysis. It was shown at the time by experimental proof that this was possible, even when the sulphate of lead was pure, and it was contended that when intimately mixed with peroxide of lead, as in actual practice, there was no difficulty about the matter[*].

---

* Read November 28, 1890.

There arose, however, a belief that a subsulphate of lead was produced on the plates. Some experimenters found in this belief an explanation of the difficulty in reducing those masses of salt which occasionally form in practice, while others looked upon the subsalt as more easily reducible than the neutral compound.

So far as we are aware, no specimen of the supposed subsalt was described till Professor Frankland published his paper in the Royal Society's Proceedings, 1889, vol. xlvi. p. 304, in which he gives an account of what he believes to be two definite subsulphates. That produced from litharge we have already examined in our paper of last June[†]. Since then we have turned our attention to the other possible subsalt made by the action of sulphuric acid on minium.

In the original paper[‡], Gladstone and Tribe gave the formula of decomposition as

$$Pb_3O_4 + 2H_2SO_4 = PbO_2 + 2PbSO_4 + 2H_2O.$$

They showed by means of a quantitative experiment its gradual slow formation, and said, "It is evident that in a Faure battery we are dealing with plates that consist of a superficial layer of mixed peroxide and sulphate of lead."

Frankland, while viewing this red substance as a subsulphate, arrived at precisely the same ultimate composition, which he expressed by the formula $S_2Pb_3O_{10}$. He adduces, however, no evidence to show that it is not a mixture of PbO_2 with $2PbSO_4$. We sought for some means of deciding between these two views. A solution of acetate of ammonium is known to dissolve sulphate of lead, and was in fact used long ago for that purpose. It might therefore be expected to separate the sulphate from the peroxide, while it was not so likely to bring about the decomposition of such a compound as $S_2Pb_3O_{10}$.

---

[*] See "The Chemistry of the Secondary Batteries" in 'Nature' Series.

[†] Phil. Mag. Aug. 1890 ,and Proc. Phys. Soc. x. p. 448.

[‡] See volume cited, p. 14.

Some of the red substance was prepared by treating minium with an excess of dilute sulphuric acid (1 of acid to 10 of water). After four days the action was still incomplete, but eventually it absorbed the full amount of acid required by theory.

The percentage of lead in the resulting red substance was just over 72 per cent.

A portion of this, after treatment with a 3 per cent. solution of acetate of ammonium, left a residue which was much darker in colour than the original and contained 82 per cent. of lead. That is to say, it was approximating towards PbO_2, which contains 86·6 per cent. of lead. The liquid was colourless, and the ratio between the Pb and SO_4 dissolved was determined in several different instances. The ratio

$$\frac{\text{Weight of Pb}}{\text{Weight of } SO_4}$$

varied from 2·0 to 2·15, pure $PbSO_4$ requiring 2·16 and Frankland's compound 3·23. The portion dissolved therefore was not a basic sulphate, and the evidence tells against the original substance being a chemical compound.

Of course, it is impossible to affirm that no subsulphate is ever formed in the reactions of a secondary cell, but all that we actually know is perfectly consistent with the old and simpler supposition that $PbSO_4$ is the salt alternately formed and decomposed.

Action of Sulphuric Acid on Lead.

In our former communication we described a comparative experiment in which dilute sulphuric acid, either alone or mixed with a small quantity of sodium sulphate, was allowed to act on spongy lead. The experiment was allowed to continue for five months, and hydrogen gas was given off for such a length of time that we thought it would be interesting to analyse the resulting substance.

The residue from the action of the pure acid on the lead consisted of 82 per cent. of sulphate of lead and 18 per cent. of metallic lead, while that from the action of acid mixed with sulphate of soda consisted of 89 per cent. of lead sulphate and 11 per cent. of metallic lead.

Although the action of the acid on the lead was initially diminished by the presence of sodium sulphate, the final result in this instance was rather the other way. No doubt the whole of the spongy lead would have been converted into sulphate if it had not been for the large amount of insoluble salt which protected the remaining particles.

This result throws light on the reason why a lead plate deteriorates during a long rest.

VI. *A Lecture Experiment illustrating the Effect of Heat upon the Magnetic Susceptibility of Nickel.* *By* SHELFORD BIDWELL, *M.A., F.R.S.*[*]

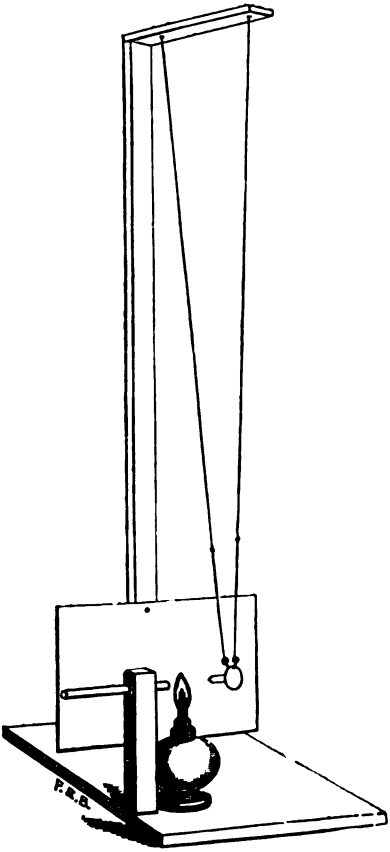
IT is well known that iron when made red-hot loses its susceptibility and practically becomes a non-magnetic metal. Nickel becomes non-magnetizable at a much lower temperature, perhaps at about 300°. The following is a description of a simple piece of apparatus by means of which this effect may be shown as a lecture experiment.

A copper disk, to which a thin projecting tongue of nickel is soldered, hangs like the bob of a pendulum from a double thread. This bob is held on one side by a horizontally-fixed bar-magnet, which attracts and holds fast the nickel tongue when brought into contact with it. A spirit-lamp is placed beneath the tongue, and in a few seconds the heat of the flame temporarily destroys the magnetic quality of the nickel, so that the magnet can no longer hold it. The bob accordingly falls back and performs an oscillation. In the course of its excursion, however, the metal becomes cooler, and when it returns to the neighbourhood of the magnet the tongue is once more attracted. But after a momentary contact it is again liberated, and the process is repeated. If the position of the magnet and the size of the flame are properly adjusted, and care is taken to shield the apparatus from currents of air, the bob may be kept swinging for an indefinite period like the pendulum of a clock.

---

[*] Read April 13, 1889.

The apparatus is shown in the annexed figure, and its dimensions are as follows:—Diameter of the copper bob 5 centim., thickness 0·08 centim. Length of the projecting portion of the nickel tongue 1·5 centim., width 0·8 centim., thickness 0·08 centim. The end of the nickel is filed to a blunt point (about 135°). A stout copper wire (0·14 centim. in diameter), to which the suspending thread is attached, passes through a hole drilled near the upper edge of the disk and is fixed by solder. The copper wire is bent in such a manner



as to make the nickel tongue hang horizontally. The copper disk and the nickel tongue, except the extreme point of the latter, are coated with a mixture of lampblack and gold size to facilitate cooling by radiation. The length of the pendulum from the suspending beam to the centre of the bob is 85 centim. In order to prevent accidents by burning, the lower part of the suspension for a length of about 10 centim. is made of fine wire; the remainder consists of ordinary sewing-silk. The round bar-magnet is about 13 centim. long and 1·2 centim. in diameter; it slides stiffly through a horizontal hole in an upright post, and its best position must be found by trial. A sheet of white cardboard is fixed behind the bob, so that the effect may be well seen at a distance.

The above dimensions might no doubt be varied considerably without detriment; but if they are followed good results may be certainly obtained without waste of time in experiment.

VII. *Alternate Current-Condensers.* *By* J. SWINBURNE*.

THOUGH condensers have been proposed for various methods of distribution of electrical power, schemes involving the use of them have never developed enough for alternating current-condensers to be made commercially. It is generally assumed that there is no difficulty in making them; all that is needed is to separate a number of thin plates by means of sheets of insulating material.

The first difficulty is insulation. If a condenser is to take 2000 effective volts, the insulation must be very good. Too great thickness of dielectric cannot be allowed, as that would lessen the capacity and increase the cost of material. The surface of such a condenser for, say, ten amperes is about a thousand square feet, and the problem of making a thousand square feet of thin insulating material that will safely stand 2000 volts is not easy. I have made samples which, though only a little over a tenth of a millimetre thick, would stand 8000 effective volts for some time before breaking down. When made up into a large condenser, two or three thousand volts would generally break it down in a few hours.

---

* Read December 12, 1890.

Insulation is not by any means the only difficulty, however; there is another, which is probably intimately connected with it, and that is absorption of electric power in the dielectric, and consequent heating. It is well known that Leyden jars get hot if charged and discharged frequently; and also that the phenomenon of electric absorption is manifested. When there is absorption, the current out of the condenser is not proportional to the rate of decrease of the electromotive force, and probably the current into it is not proportional to the rate of increase. This means that the condenser absorbs energy and converts it into heat.

Such questions have been frequently studied in connexion with curves of sines, but it is easier to show the truth of this generally. Let C be the current, E the electromotive force, K the capacity, and t the time. Then

$$C=K\frac{dE}{dt},$$

and the power spent at any instant,

$$EC=EK\frac{dE}{dt}.$$

If e is the maximum electromotive force, the work done in charging the condenser is thus

$$K\int_0^e E\,dE=\tfrac{1}{2}Ke^2,$$

the ordinary expression. On discharging, the work absorbed is

$$K\int_e^0 E\,dE \quad \text{or} \quad -\tfrac{1}{2}Ke^2.$$

It is equally easy to show that no power is absorbed in the case of an induction-coil with no hysteresis, without assuming the pressure to vary harmonically.

If C does not vary as the rate of increase or decrease of E but is greater on charging and less on discharging, power is absorbed. A hysteresis curve, like those we are accustomed to in iron, can be plotted in the case of a condenser. It has no very close analogy with the case of iron, however.

At first it might seem that absorption might be neglected as too trifling to matter in commercial work. This, however, is not the case. Some condensers about a foot square and an inch thick absorbed over half a horse-power, and soon rose above the temperature of boiling water.

According to Maxwell's theory, the absorption of power is just as easily explained as the time-lag in discharge. Suppose the dielectric is paper soaked in melted paraffin, and suppose the fibres of paper do not insulate and that the paraffin does. When the plates are charged, a fibre running part of the way across the dielectric finds its ends at different potentials ; so a current is set up to equalize them. This current means loss of power by heat. It also increases the capacity of the condenser. An increase of the capacity of the condenser when the pressure is increasing, or after it is applied, means increased energy put into the condenser. If the capacity of the condenser were constant, the current into it would be proportional to the rate of increase of the electromotive force, and no power would be absorbed in a period ; but if the capacity is increased as the pressure rises and decreased as it falls, there is an extra current produced which is not proportional to the rate of increase of the electromotive force, and this supplies the loss in the condenser. The condenser may still show perfect insulation under a direct pressure. According to this theory, absorption is necessarily accompanied by an increase of capacity. This is important in connexion with the electromagnetic theory of light. Absorption would always increase the apparent capacity of a condenser, so that determinations of specific inductive capacity of absorptive dielectrics would come out too high. The dielectrics which come out too high, such as glass, are just those which heat most. The energy absorbed in a given dielectric would be a function of the frequency and the resistance of the conducting parts. It is thus possible that a dielectric might be opaque to light, and diathermanous if the resistance is comparatively low, or transparent to light, but not to dark heat if it is somewhat higher. Similarly a Leyden jar, which absorbs power when connected to an alternating dynamo, may lose little of its energy internally when discharged oscillatorily by a spark. The case of a fibrous imperfect insulator embedded in a perfect insulator is not susceptible of mathematical treatment and was not taken by Maxwell. Mica is perhaps a better example. It is a good insulator, but gets very hot. If a thin plate is put between two sheets of tinfoil, heating soon shows. Here we have tinfoil, then a stratum of air, then mica, which is itself stratified, then air, then tinfoil again. The static attraction presses the foil close against the mica, still there is air. Unless the mica is very thick, the fall of potential between the plates is so rapid that the air breaks down and there is a disruptive discharge between the foil and the mica. This actually occurs; there appears to be a luminous layer of minute blue sparks under the foils, and there is a strong smell of ozone. The discharge under the foils, which is rendered visible by the transparency of the mica, must not be confused with the brush-discharge round the edges of the foil, which differs in appearance. There is considerable heating even when the foil is affixed with paraffin-wax so that there is no air.

Of course mica, being stratified, may consist of alternate sheets of some imperfect conductor and an insulator. It is more difficult to find a reason why glass should heat. At high temperatures it is an electrolyte, and it has no definite solidifying-point, so it may be partially electrolytic at ordinary temperatures. Such an explanation can hardly be applied to such a substance as celluloid. This is a very perfect insulator, at least till it breaks down disruptively, and it seems to be homogeneous, and is not hygroscopic, yet it heats considerably.

For commercial condensers the choice of insulating material is limited. Most work has been done on condensers with paper and hydrocarbon insulation. It is difficult to know whether a fault lies in the paper or in the hydrocarbon. The paper is baked at a high temperature, and all temperatures and times of baking have been tried. Paper goes on giving off water till there is nothing but a charred brittle mass left. Many hydrocarbons contain enough water to prevent, for instance, their dissolving rubber. Phosphoric anhydride was used to dry several samples, but seemed to do little good, in fact the crude hydrocarbon was often better. Papers vary astonishingly in their behaviour. Mr. Bourne, who has been working at these things for some months, has tried almost every conceivable way of making condensers. We are just beginning to succeed, but it is more by chance than anything else. We see no reason why one particular kind of paper and one particular kind of hydrocarbon should be better than the others.

Want of homogeneity in the dielectric of a condenser, or of any cable, may lead to disruptive discharges, or breakdowns. For instance, if a dielectric is made up of portions with inductive capacities of 3 and 1 respectively, the parts with a high inductive capacity may arrange themselves so that the "electric displacement" in the other is so great that a disruptive discharge ensues. For instance, if two conducting plates are 3 millim. apart in air, with enough pressure to spark over 2 millim., and if a 2-millim. slab of a dielectric with a specific inductive capacity of 3 is put in, the fall of potential over the air is nearly doubled, and it breaks down and starts a short circuit.

Loss of power in dielectrics is not confined to condensers; it may be very serious in cables. A condenser that heated excessively was made like the Deptford mains. The dielectric was very much thinner in proportion to the pressure, and allowing the loss per cubic centimetre to vary as the square of the pressure on its sides, a Deptford main would have a loss of, very roughly, 7000 watts in the dielectric. It does not follow that this is the exact loss in a Deptford main. Very small differences in the constitution of the dielectric cause large variations in tho power wasted. The loss may therefore be very much greater or very much less. Taking the specific conductive capacity of the dielectric as 2, a seven-mile main has a capacity of 2 microfarads. With 10,000 volts at a frequency of 80, this takes 10 amperes and 100,000 "apparent" watts; 7000 watts is only a small percentage of this.

Another unexpected effect has occurred at Deptford. There is an extraordinary rise of pressure. No authoritatively accurate accounts have been published, so it is difficult to say what has happened; but it is said that when the mains are put on the pressure rises. This is generally explained by saying the mains have capacity, and there is self-induction in the circuit, and the capacity and self-induction have a period which corresponds with the frequency of the dynamo, so that the system sympathises, or acts as a sort of electric resonator, thus giving abnormally high pressure. A little consideration will show that this theory is untenable. With such a capacity as that of the Deptford cables the self-induction would have to be enormous.

Let the capacity of the cable be K, and the self-induction of the circuit L, in farads and henries or quadrants. Call E and C the electromotive force and current respectively. Then

$$E=L\frac{dC}{dt}$$

and

$$C=-K\frac{dE}{dt}.$$

Combining these equations,

$$\frac{d^2E}{dt^2}=-\frac{E}{KL};$$

so

$$\frac{dE}{dt}\frac{d^2E}{dt^2}=-\frac{E}{KL}\frac{dE}{dt};$$

and integrating,

$$\left(\frac{dE}{dt}\right)^2=-\frac{1}{KL}(E^2+a),$$

where a is a constant;
or

$$\frac{dE}{dt}=-\frac{1}{\sqrt{KL}}\sqrt{(E^2+a)};$$

so

$$\frac{\sqrt{KL}dE}{\sqrt{E^2+a}}=-dt,$$

and integrating,

$$\sqrt{KL}\cos^{-1}\frac{E}{a}=t,$$

therefore

$$E\propto\cos\frac{t}{\sqrt{KL}}$$

and

$$T=2\pi\sqrt{KL}$$

When a charged condenser is short-circuited through a circuit with self-induction only, oscillation is produced, the current and pressure both varying harmonically with a frequency of $1/2\pi\sqrt{KL}$.

In the Deptford case, with a capacity of two microfarads and a period of 0·0125, the self-induction needed would be half a henry. A choking coil with such a coefficient of self-induction wound to carry 250 amperes, which, I understand, is the load of one main, would take 64,000 volts to get the current through. A transformer only acts on the circuit as a choking coil to the small extent due to the waste induction in it. If built on the lines of commercial transformers, this one would have to be large enough to give an output of hundreds of millions of watts to produce a resonator effect. Moreover, a slight alteration in the speed of the dynamo would throw it "out of tune" with the resonator, so that the effect would disappear. For instance, a 5 per cent. variation of speed would alter the pitch of the dynamo nearly a semitone.

It has been stated that there is a difference of apparent ratio in the Deptford transformer when the main is in circuit; that is to say, that though it generally transforms 4 to 1, when the main is on it transforms about 4½ or 5 to 1. I would suggest this is impossible. There is always a "drop" in transformers due to waste field. In a transformer for 10,000 volts and 250 amperes, if properly designed, it would be well under 1 per cent., probably about one tenth per cent. In order that an oscillatory current should be confined to the "line" side by the waste induction which causes a drop of 1 per cent. and 80 periods per second, the frequency would have to be enormous. Such an oscillatory current would also be excluded from the secondary at the London end. Any effect which does not involve enormous frequencies must show on both sides of the transformers. It might be said that the "drop" of the transformers in the circuit would give enough self-induction to produce a frequency corresponding to an upper partial of the note of the dynamo. If the dynamo E.M.F. does not vary harmonically, such an effect might be produced, but it would show on the voltmeters at both ends. To give an increase of 15 or 20 per cent. effective pressure, with a total transformer drop of 1 per cent., the frequency would be, roughly, 1000, and the effective pressure of the oscillatory component 6000 volts. This would give 72 amperes, and the "skin" effect would absorb a great deal of power.

There is, however, a simpler explanation. The cables have considerable capacity, and take an appreciable current. This "leads" relatively to the electromotive force, so the capacity current passes in the armature-coils when they are just in the position to magnetize the fields more strongly. The fields cannot follow each pulsation of excitation so produced, but are affected by it. Their average excitation is increased. I have tried putting a condenser on one of the whole Gramme alternators, which have rather weak fields, and a large number of armature ampere turns. The pressure ran up and burned the voltmeter. I have already gone fully into the action of leading and lagging currents on dynamo fields elsewhere, so that subject need not be pursued here. It would, no doubt, be possible to make an alternator excite itself like a series machine by putting a condenser on the terminals instead of exciting the fields by a direct-current machine. Such an arrangement seems scarcely commercial, though interesting.

VIII. *The Solution of a Geometrical Problem in Magnetism.*

By T. H. Blakesley, *M.A.*[*]

The points in the field of a magnet usually chosen for quantitative experiments, such as the determination of H, or the control of a galvanometer-needle, lie either in the axis of the magnet produced, or the equatorial plane. This arises from the very simple expressions for the field in terms of the moment of the magnet, and its distance from the point considered, in these two cases. But in either of these cases the exact value depends not merely upon these facts, but also upon the distance between the poles; and this latter can rarely, if ever, be taken to be the entire length of the magnet. Either some such rule as the lopping-off, in imagination, of a fraction of the length, is applied, or the virtual distance between the poles is expressed as an unknown quantity to be determined by additional experiment.

---

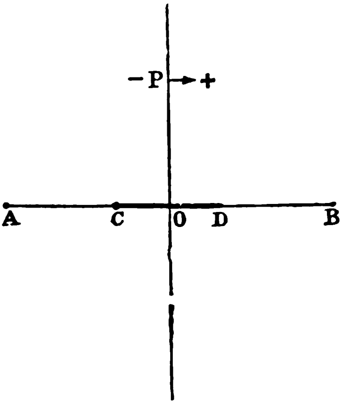
[*] Read November 28, 1890.

It is easy also to calculate the direction and magnitude of the field at a point whose position relatively to the two poles is given. But the following proposition is, at first sight, of a more difficult order.

"Given the two poles of a magnet, and a straight line intersecting at right angles its axis produced at a given point, to determine at what point this line is parallel to the field."

The solution of this question may be of some scientific interest; because if we know the point experimentally, we can determine the length between the virtual poles. But the question is important practically from its bearing upon that of the deviation of a ship's compass in some cases.

Fig. 1.



Suppose A B a long uniformly magnetized rod having poles at A and B (say A is a north pole), and O P its equatorial plane. Then the field at P is always parallel to A B, and in the direction of those letters, which call the positive direction. Suppose a piece C D cut away so as to leave a gap having same equator.

At C a south pole is developed, and at D a north pole.

These two poles produce a field at P parallel to D C, and in the negative direction.

If the point is in the neighbourhood of O, the two inner poles will be dominant and the field negative; but if P is very remote from O, the poles A and B will be dominant, and the field positive. There must therefore be some position where the field vanishes, and the two rods A C, D B would produce no deviation on a compass at that point, in whatever direction the head of the ship carrying the system pointed.

Such permanent magnets would produce what is called semicircular deviation on a compass situated at any distance but this critical one from O; that is, through one semicircle the deviation would be easterly, and through the remaining semicircle westerly; but these semicircles would each have the deviation produced in it changed in sign if this critical point is transgressed. If, however, the rods A C, D B are of soft iron, liable to magnetization by the action of the horizontal component of the earth's field, the general effect is that the direction of the deviation changes sign after every *quadrant*, each pair of *opposite* quadrants having one kind of deviation, easterly or westerly; and in this case the actual sign for each pair of opposite quadrants depends upon the position of the compass relative to this critical position.

This application may serve to excuse me for bringing forward an easy method of practically finding this point.

It is clear that the point is one for which the field due to either magnet, A C or D B, alone, would be entirely along O P, *i. e.* at right angles to the axis of the magnet.

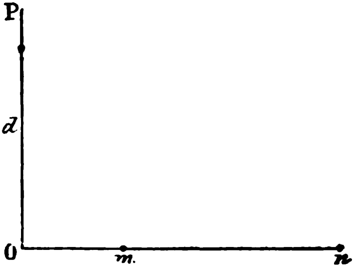
The Solution.

Let $n\,m$ be the position of the poles of a magnet, and P a point situated at distance d from O, O being in the axis of the magnet produced, and P O being perpendicular to O $m\,n$.

Let the distance

$$On = n.$$
$$Om = m.$$

Fig. 2.



Let p be the numerical value of the poles m and n.

Then, writing down the condition that the component parallel to the magnet of the field produced at P by p at m shall be equal and opposite to the same thing produced by p at n, we obtain

$$\frac{m}{(d^2+m^2)^{\frac{3}{2}}}=\frac{n}{(d^2+n^2)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as the equation to find d, all the other measurements being known.

Manipulating this expression, we have at length

$$\left(\frac{d^2}{2mn}\right)^3-\frac{3}{4}\times\frac{d^2}{2mn}-\frac{1}{4}\times\frac{m^2+n^2}{2mn}=0. \quad . \quad . \quad (2)$$

Now it happens that in hyperbolic trigonometry we have

$$\cosh^3\theta - \tfrac{3}{4}\cosh\theta - \tfrac{1}{4}\cosh 3\theta = 0. \quad . \quad . \quad . \quad . \quad (3)$$

If, therefore, we make

$$\frac{m^2+n^2}{2mn} = \cosh 3\theta, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have also

$$\frac{d^2}{2mn} = \cosh\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By means, therefore, of the table of hyperbolic sines and cosines which this Society has recently published, we can easily determine *d*.

We can deduce the distance of the poles apart, by applying this proposition inversely.

It is of course very easy to arrive at this state of things practically with a magnet.

Suppose we arrange a small magnetometer-needle in the meridian, and notice the position of the light-beam on a scale in an ordinary way.

Find the direction of the axis of the needle by the condition that a long magnet laid in that line will not affect the position of the spot of light; then place that magnet, or any other, at right angles to this position and move it in the direction of its length until again the spot of light is at its old place. No torsion affects these observations; indeed, the real meridian need not have been the direction found.

Then, obviously, of the various quantities quoted in the problem, we are in direct possession of d and $m+n$ (the latter because the middle point of the magnet is distant $\frac{m+n}{2}\,(=l)$, perpendicularly from the axis of the magnetometer-needle); but the pole-distance $m-n=2a$ is only implicitly known.

Its value may be deduced from the equations (4) and (5); thus

$$\frac{m-n}{m+n} \text{ or } \frac{2a}{2l} = \frac{a}{l} = \tanh\frac{3\theta}{2},$$

where

$$\frac{l}{d} = \sqrt{\frac{\cosh 3\theta + 1}{4\cosh\theta}}.$$

The latter function is not hard to deduce from the tables.

Experiments on this plan showed that the virtual distance between the poles soon approaches the full length of the magnet.

It remains to give the expression for the field under the circumstances. It is

$$\frac{4\mathrm{M}}{d^3}\frac{\cosh^2\theta}{(4\cosh^2\theta-1)^{\frac{3}{2}}},$$

where M is the moment, or $p(n-m)$.

It will be noticed that we have $\frac{l}{d}$ at our disposal if we allow d to vary, *i. e.* one degree of freedom. Suppose, therefore, we arrange to simplify the expression for the field by putting

$$\cosh^2\theta=\frac{5}{4}.$$

Then the field $=\frac{5\mathrm{M}}{8d^3}$;

but at the same time, since

$$\cosh\theta=\frac{\sqrt{5}}{2},$$

$$\cosh 3\theta=\sqrt{5},$$

$$\therefore \frac{l}{d}=\sqrt{\frac{\cosh 3\theta+1}{4\cosh\theta}}=\left(\frac{5^{\frac{1}{2}}+1}{2\times 5^{\frac{1}{2}}}\right)^{\frac{1}{2}}$$

$$=\left(\frac{1}{2}+\frac{\sqrt{5}}{10}\right)^{\frac{1}{2}}$$

$$=\cdot 850651.$$

Let $\tan\phi=\frac{l}{d}$, then

$$\phi=40^\circ\ 23'\ 10'',$$

and we must keep the middle point of the magnet on the line making this angle with the meridian.

IX. *Some Experiments with Selenium Cells.*

By SHELFORD BIDWELL, *M.A.*, *F.R.S.*[*]

IT is well known that selenium, like sulphur, which it closely resembles in its chemical properties, is capable of existing in several distinct forms, differing greatly in appearance. The modifications most commonly met with are known as the black amorphous or vitreous, the red amorphous, and the crystalline forms. Selenium is, I believe, always supplied commercially in the vitreous condition. Vitreous selenium is as structureless as glass, and, when its surface is smooth and clean, it is very like black sealing-wax in appearance. In thin films, however, it is seen to be transparent and of a ruby-red colour. It melts, after first softening, at a temperature of about 100° C. Red amorphous selenium is a finely divided brick-red powder, which, at a temperature of from 80 to 100°, becomes transformed into the ordinary dark-coloured vitreous form. Crystalline selenium is obtained by keeping melted vitreous selenium at a temperature of from 100° to 200° for some time, the black liquid mass being gradually converted into a hard slate-coloured solid. This form of selenium melts at 217°: it resembles a metal in appearance, and even in the thinnest films it is quite opaque to light.

In the vitreous condition selenium is a practically perfect non-conductor of electricity: so it is also in the powdery condition if perfectly dry. Crystalline selenium, on the other hand, is, according to the books, a good conductor. In comparison with true metals, however, it conducts rather badly. I found some time ago that the specific resistance of a plate of crystalline selenium, which had been annealed for several hours in a glass mould, out of contact with any metal, was 2500 megohms, while that of another specimen annealed in contact with copper electrodes was 0·9 megohm, the lower resistance in the latter case being due, as I then suggested and still think probable, to the formation of copper selenide during the heating.

A very remarkable fact, first published by Mr. Willoughby

---

[*] Read December 12, 1890.

Smith in 1873, is that the resistance of crystalline selenium appears to be temporarily diminished by the action of light. This effect is easily shown by means of a small piece of crystalline selenium fitted with wire electrodes and joined up in circuit with a battery and a reflecting-galvanometer. The galvanometer-deflexion is increased when the selenium is illuminated.

In order that the resistance of a so-called "selenium cell" may be kept as low as possible, and the light made to produce a maximum effect, three conditions have to be aimed at. The electrodes should be large; they should be near together; and the selenium, in order to expose a large surface relatively to its volume, should consist of a thin film. Several contrivances have been proposed with the view of fulfilling these requirements. A simple and, I believe, effective one is that which I suggested ten years ago ('Nature,' vol. xxiii. p. 58). A fine copper wire is wound upon an oblong strip of mica from end to end, the number of turns being about 20 to the inch: a second wire is then wound on, parallel to the first, the turns of the second wire alternating with those of the first; great care is taken that the two wires do not touch each other anywhere. A thin layer of melted selenium is spread over one surface of the mica, filling the spaces between the wires. The film is first quickly cooled by placing the mica upon any convenient cold surface, then it is crystallized by heating at a temperature of about 200°, and, lastly, it is annealed by gradually lowering the temperature for several hours. A cell made in this manner, with a surface of about 10 sq. cm., is generally found to have a resistance in the dark of from 50,000 to 100,000 ohms. As a rule, those cells which have an unusually low resistance are less sensitive to light than others. In a cell of average quality the resistance is reduced about 50 per cent. by the light of an ordinary gas-flame at the distance of one foot.

I made a number of cells on this plan in the years 1880 and 1881. Up to the end of 1882 they were frequently used, and no material falling off in their sensitiveness was noticed. Thirteen of these cells were laid aside until 1885. They had then for the most part deteriorated more or less, one or two of them being quite useless. They were again put away until the present year, 1890, when it was found that only one of the thirteen retained its original sensitiveness and resistance unimpaired.* A second one worked fairly well with a feeble current, its resistance with a single Leclanché cell being in the dark 14,000 ohms, and when illuminated by a gas-flame at 12 inches, 8000 ohms; but with a battery of six cells the resistance became variable and unsteady, as if there were a loose contact somewhere, and a continuous boiling noise was heard in a telephone introduced into the circuit. The resistance of the remaining eleven cells had fallen enormously, in some cases to less than 10 ohms, and they were quite insensitive to the action of light. An attempt was made to restore most of them by melting down the selenium and recrystallizing and annealing; but though the resistance could in this manner be brought up again to a high point, the sensitiveness of the renovated cells was in all cases found to be very poor.

Prof. Adams states (Phil. Trans. vol. clxvii. p. 348) that in the course of a year the resistance of most of the pieces of selenium with which he performed his well-known experiments fell very considerably, and it would be of great interest to know what is the cause of this curious phenomenon. In the case of my own cells, I satisfied myself that it certainly was not due to any short-circuiting of the wires by contact; and I suggest that it may be owing to the presence of an excessive amount of conducting selenide formed by the union of the selenium with the metal of the electrodes. Each wire is covered with a layer of selenide, which gradually increases in thickness at the expense of the free, badly-conducting selenium, and thus the resistance of the cell slowly falls. At length the layers of selenide become so thick that the coatings upon the two wires actually meet and touch each other. Thereupon the cell is short-circuited; its resistance no longer depends upon the selenium but upon the metallic selenide, which is a good conductor.

I have in a former paper (Proc. Phys. Soc. vol. vii. p. 129; Phil. Mag. Aug. 1885, p. 178) given reasons for believing that the presence of a certain amount of metallic selenide is essential to the sensitiveness of a cell, and I have stated my opinion that the true function of the prolonged heating, generally termed "annealing," is the formation of this necessary selenide, which, when a current passes through the cell, is electrolysed, selenium being deposited upon the anode wire. With the weak currents generally used, the quantity of deposited selenium would be very small, and would at once unite with the metal of the electrode. But if a sufficiently strong current were passed, it is conceivable that the selenium might be separated in quantities too great to be disposed of in this manner. Then the free selenium around the electrode ought to be visible. This view is confirmed by a recent experiment. A cell was made last October with new selenium and the usual copper-wire electrodes. It was well annealed, and its resistance was about 50,000 ohms. The cell was connected to a 26-volt battery, and left for two days. At the end of that time the cell presented a remarkable appearance, those portions of the anode wire where the selenium coating happened to be thin being covered with red feathery tufts, some of which had dropped off and fallen upon the paper on which the cell was lying. Ordinary tests showed that the red stuff consisted, mainly at all events, of red amorphous selenium. But it was clearly moist. All very fine powders seem to attract moisture from the atmosphere ; but in the present case the moisture appeared to be greater than could be thus accounted for. An examination under the microscope disclosed the presence of a number of minute lumps of a white substance, intermixed with the red stuff. The lumps had the appearance of pieces of calcium chloride which had been exposed to the air, and they were clearly the origin of the moisture. They consisted, no doubt, of selenium oxide or hydroxide, which is probably a deliquescent substance.* This oxide might perhaps be formed by the direct combination of some of the freshly separated selenium with the oxygen of the air, or with that resulting from the electrolysis of traces of water existing in the cell.

---

* This cell had the unusually high resistance of 400,000 ohms, reduced by the light of a gas-jet at one foot to 190,000 ohms.

In the discussion which followed the reading of my former paper, the late Mr. J. W. Clark took exception to a hypothesis therein put forward as to the probable action of the selenides formed in selenium cells, on the ground that selenide of copper conducted metallically and not electrolytically. This he stated to have been experimentally proved by Hittorf (Pogg. *Ann.* vol. lxxxiv. 1857, quoted by J. W. Clark, Proc. Phys. Soc. vol. vii. pp. 119, 120 *; Phil. Mag. July 1885, pp. 38, 39). But if the amorphous selenium surrounding the anode in my recent experiment was formed otherwise than by electrolysis, it is difficult to imagine why it should be confined to the anode, as was seen to be the fact, and not extend also to the kathode and other portions of the cell. The inference seems to be that copper selenide does conduct electrolytically †.

---

* Since the above was written, Prof. S. U. Pickering, F.R.S., has told me that the oxides of selenium are deliquescent.

When the resistance of this cell was tested with the Wheatstone's bridge, it was found to have diminished enormously. It was also not sensibly affected by illumination, but, on the other hand, it appeared to be dependent in a curious way on the direction of the current used in testing. With one Leclanché cell and equal proportional coils of 100 ohms each, the resistance with a current from anode‡ to kathode was 4500 ohms, and with a current in the opposite direction only 2900 ohms. The change occurred with great regularity as often as the current was reversed.

The cell was then connected directly to the galvanometer without any battery, and a very small polarization current from kathode to anode through the cell was indicated. While thus connected, a piece of magnesium ribbon was burnt near the cell. Instantly the spot of light was deflected off the scale by a current in the opposite direction (*i. e.* from anode to kathode internally), returning to its former position as soon as the magnesium was extinguished. This was not a thermo-electric effect, for the approach of a hot brass rod caused a smaller current *from kathode to anode*, which subsided slowly and not suddenly when the rod was withdrawn.

---

* This paper of Mr. Clark's, though printed in the 'Proceedings' before my own, was in fact read after it.

† The cell with the red amorphous selenium on the anode was exhibited to the meeting.

‡ The wire upon which the amorphous selenium was deposited is here and afterwards called the anode.

Photoelectric currents of this character had been previously observed by Prof. Adams and by Mr. Fritts of New York, but no explanation of their origin has been attempted. Perhaps the behaviour of my cell may suggest the direction in which the source of such currents should be looked for, but I have not yet followed up the inquiry.

In the illustration of lectures it is often desirable to be able to exhibit experimental effects in a striking and attractive manner. The movements of a spot of light upon a scale may be convincing, but they are hardly impressive. Yet the resistance of sensitive selenium cells is so high that the variation, under the influence of light, of currents passing through them cannot easily be demonstrated to an audience in any other manner than by their action upon a reflecting-galvanometer. With sufficient battery-power they may, however, be made to work a delicate relay, and thus to produce indirectly a great variety of startling effects. For example, a selenium cell may be connected with a relay and a battery in such a manner that a bell is rung when the cell is in the dark and ceases to ring when the cell is illuminated; the bell circuit remaining closed when the current through the selenium circuit is below a certain strength, and being broken by the relay when the selenium current is increased by the action of light. It is possible that if selenium cells could be made to retain their sensitiveness for a reasonable time, an arrangement of this kind might receive a practical application. It might be used to give notice of the accidental extinction of railway signal-lamps or of ships' lights; and if the connexions were so made that light and not darkness set the bell in action, it might be employed for the protection of safes and strong-rooms, the mere light of a burglar's lantern being sufficient to give an alarm.

Again, if the relay is connected with an electric lamp instead of a bell, it constitutes a contrivance by means of which a light is provided automatically when required, and extinguished when no longer wanted. It gives us a lamp which will light itself in the dark and put itself out in the light. The relay can be adjusted so that the lamp may be turned out when the external illumination reaches any desired point, the diffused light of a dull December day being quite sufficient to act upon the selenium [*].

But I do not at present attach any serious importance to such practical applications of these devices. I regard them simply as affording somewhat attractive illustrations of the effect of light upon the resistance of selenium.

X. *Experiments in Photoelectricity.* *By* GEORGE M. MINCHIN, *M.A., Professor of Mathematics in the Royal Indian Engineering College, Coopers Hill* [†].

NEAR the beginning of the year 1877 I commenced a long series of investigations in Photoelectricity, and I was not then aware that M. Becquerel had previously worked at this subject and had obtained some results which I also obtained independently. In the spring of 1880 I made a verbal communication to this Society, illustrated by experiments, on the generation of electric currents by the action of light on silver plates which were coated with collodion and gelatine emulsions of bromide, chloride, iodide, and other salts of silver, as well as with eosine, fluorescine, and various aniline dyes. A short summary of these results appears in the Report of the meeting of the British Association that year at Swansea. The chief object which I had then in view in prosecuting such experiments was the solution of the problem of producing a photographic image of an object at a distance by means of a instrument which is still imaginary and which I proposed to call a *telephotograph.*

As these experiments have never been published in adequate detail, I propose to begin this paper with an account of them.

Silver Plates.

Let two strips of cleaned silver foil be each fastened on a plate of glass, by means of pitch or other suitable substance; take some finely powdered chloride of silver and shake it up well in a test-tube containing collodion, the emulsion being, of course, screened from light by a covering of black paper on the test-tube; pour a thin coating of the emulsion over both silver plates in the dark; and when the films have set, immerse both plates in a glass vessel containing distilled water with a few grains of common salt; screen one plate completely; and then connect the plates with the terminals of a Thomson galvanometer. A disturbing current will, of course, be produced; but if the plates are very nearly alike, this current will be small and will soon almost disappear.

---

[*] Experiments with the bell and the lamp were shown at the meeting.

[†] Read January 16, 1891.

Now, on exposing the unscreened plate to light, a current will be produced, and the exposed plate is *negative* to the unexposed (*i. e.*, as copper to zinc).

The direction of the disturbing current is not in any way related to the direction of the current produced by light—as is, indeed, *à priori* evident.

If the plates are coated with an emulsion of bromide of silver, the liquid being distilled water with a few grains of bromide of potassium, the exposed plate is made also *negative* with respect to the unexposed by the action of light.

If the plates are coated with iodide of silver by first pouring a layer of iodized collodion over them, and then immersing them in a nitrate of silver bath, the liquid being water with a few grains of iodide of potassium, we obtain a reversal of the nature of the exposed plate; *i. e.* the action of light makes the exposed plate *positive* with respect to the unexposed.

By placing coloured glasses in front of the exposed plate in each of these cells, it was found that the red rays produced comparatively feeble currents, while the currents produced by the blue and violet rays were very great; but the directions of the currents were the same for all rays.

A somewhat different result was obtained when the plates were coated with an emulsion of sulphide of silver, the liquid being water with a few grains of sulphate of potash. The exposed plate is, as in the case of iodide of silver, *positive*. In this cell the directions of the currents were the same for rays passing through all the coloured glasses; but the strength of the current was very much less for rays passing through the green glass than for the rays at each side of the green.

When the plates were coated with an emulsion of nitrate of silver and gelatine, the water containing a few grains of nitrate of barium, the exposed plate was positive, the effect of the red rays being exceedingly small and that of the blue very great.

Photographic Effect of the Current.—A fact of very great importance in this subject is that which relates to the action of an electric current which passes through a photographic layer on a plate.

Two silver plates, each coated with a film of Liverpool emulsion (bromide of silver), were immersed in a tumbler containing distilled water and a few grains of bromide of potassium. One of these plates was connected with the zinc and the other with the carbon pole of a bichromate cell. The current was allowed to pass through the plates for a few seconds, with the following results :—

(1) The plate connected with the carbon pole was, without the employment of any developer, visibly blackened in its immersed part.

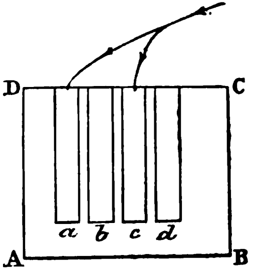
(2) No *visible* change took place on the other plate ; but when the plate was developed, by pouring over it the usual pyrogallic acid developer, its immersed portion was also blackened.

The photographic result was also obtained when the bichromate cell, which originated the current, was replaced by a photoelectric cell exposed to light. A vessel containing distilled water and a small quantity of bromide of potassium was placed in a dark room; in this vessel were immersed (or partially immersed) two silver plates S′, V′, the first coated with a film of Liverpool emulsion, the second uncoated. These plates were connected by insulated wires with the plates, S, V, of a glass cell containing water and a small quantity of common salt, the plate S being coated with a layer of a chloride of silver emulsion, and the plate V being uncoated. Magnesium light was then allowed to fall on the plate S of this latter cell for a few minutes. Now, whether S′ was connected with S or with V, the plate S′ when taken out of the vessel and developed was very sensibly blacker on its immersed portion than on the unimmersed. The action on the plate S′ was assisted by exposing it for about ten seconds to gaslight before the photoelectric current of the exposed cell was passed through it. The effect is undoubtedly due to the action of the current, because when the plate S′ was left immersed in its cell for twenty hours and then developed, no blackening effect was produced.

Not only is this photographic effect of the current important, but it is also important to prove that the effect on a sensitized plate is strictly confined to those portions of it through which a current passes; and to establish this fact more fully the following experiment was made :—

On a glass plate, ABCD, were cemented, close together but not in contact, several strips of silver foil, *a*, *b*, *c*, *d*, . . ., the whole plate being uniformly coated with a film of Liverpool emulsion; two or more of the silver strips, *a*, *c*, were put into the circuit of a bichromate cell, the other strips, *b*, *d*, being left out of the circuit. This glass plate with its strips was half immersed in a vessel of water in presence of another silver plate, and the current was passed for a few seconds. On removing the sensitized plate from the vessel and applying the developer, the blackening took place only on those strips which were metallically thrown into the circuit, *and only on the immersed portions of those strips.*

Fig. 1.



I made use of this principle in an attempt to transmit an image of a simple figure to a distance; but the arrangements were so difficult that the success attained was small, and I must leave the matter for renewed trial.

Eosine.—Comparatively strong currents are obtained by coating a silver plate with an emulsion of eosine and gelatine, and the currents are strengthened by allowing the film to set thoroughly on the plate. But such a plate has the drawback that the eosine readily leaves the film and comes into the liquid. This passage of the eosine may be delayed by pouring a layer of collodion over the dry gelatine film. By the action of light, this plate is rendered *negative* to the unexposed plate.

When daylight was allowed to fall on this plate, any variation of the light caused by a passing cloud, or the interposition and withdrawal of the hand in front of the cell, was at once accompanied by a variation of the current strength; and the same thing is true for all the cells previously described; but no variations of sufficient rapidity are produced to affect a telephone, although the make and break of the current itself is, of course, amply sufficient to do so.

The photographic action on a bromide-of-silver plate placed in a cell in a dark room and connected with one of the poles of an eosine cell was effected by the current generated by daylight in the latter without any preliminary exposure of the bromide plate to gaslight.

A very curious case of *inversion of the current* produced by light was observed in a cell containing eosine in solution; and as the same effect was occasionally observed in cells of other kinds employed in experiments made long afterwards, I shall draw attention to the phenomenon here.

Two clean silver plates were immersed in a glass cell containing distilled water and a very small quantity of eosine. One of these plates was screened and the other exposed to light, both being connected with the galvanometer. Immediately on exposure of the plate to light, there was generated a current in which the exposed plate was *positive* to the other. This current, however, lasted for only a second, and it was then (the exposure continuing) succeeded by a steady and much stronger current in the opposite direction, this latter being the current which would exist if the plate had been coated with an eosine-gelatine film in the usual way, any variation in the intensity of the light being answered by a corresponding variation in the strength of the photoelectric current. Now when the light was suddenly shut off from the plate, the instantaneous effect was to *increase* the existing current—the effect being merely impulsive—after which (the plate being screened) the current gradually disappeared. This result was again and again reproduced, and exactly the same result was found if the water in the cell contained a small quantity of fluorescine instead of eosine, except that the initial and final impulsive currents were much smaller with fluorescine than with eosine.

The plates of this cell having been left in the liquid and kept in the dark for a fortnight, the action of light was again tried; and it was then found that, while the inverse currents were produced as before, the initial current on exposure was enormously increased both in magnitude and in duration. It now disappeared gradually, and was succeeded by a current in the reverse direction.

When one of these plates was removed from the cell and immersed in water in presence of a clean silver plate, it was at once, on exposure to light, *negative*, like a silver plate coated in the ordinary way with an emulsion of eosine.

M. Becquerel, in the course of his experiments on the electric action of light on plates coated with salts of silver, made the observation that in the case of *silver* plates coated with bromide, chloride, and iodide of silver, the nature of the exposed plate (whether positive or negative) depends on the *thickness* of the layer deposited on the surface of the plate. Thus, he says (*La Lumière*, vol. ii. p. 129):—"By depositing on one of these plates a *thin* layer of iodide, obtained by the action of the vapour of iodine at the ordinary temperature, and then exposing this plate to light, it was found that it took *positive* electricity from the liquid. . . . . With a *thick* layer of iodide on the surface of the silver, there is, on the contrary, a current the inverse of the preceding; that is to say, the plate exposed to light took *negative* electricity. This result shows that, in this case, iodine acts on silver under the influence of light." He then gives a table of deflexions obtained when various coloured glasses were placed in front of a silver plate coated with a thick layer of iodide, the greatest effect being produced by violet rays, and the least by red, the former being 22 times the latter.

M. Becquerel continues:—"Thus, whilst with chloride and bromide of silver precipitated, placed on plates of platinum, there is always produced a current of the same sense, the exposed plate being positive, with a layer deposited on silver, there is an effect depending on the thickness of this layer. These two inverse effects indicate that there should be necessarily a thickness for which the electric effect is almost nothing."

These observations of M. Becquerel may possibly explain the above inverse effects in the eosine-silver cell; but his last conjecture would not be practically verifiable, unless, when the critical thickness was reached, the state of the sensitive surface was one of stability. One result, however, would pretty evidently follow, viz., that if on the same plate a portion of the layer was below the critical thickness, while the remainder was above it, the plate, on exposure, might be, on the whole, positive, negative, or inert. In the case of photoelectric cells to be subsequently described, it is possible that the observed results are due to this cause.

Certain mordants were tried for the purpose of preventing the eosine from leaving the film. The sensitized plate was washed with a solution of borax, which had the effect of keeping the film on the plate, but almost completely destroyed its sensitiveness to light. A similar effect followed from the employment of a solution of chloride of aluminium. The best effect was produced when a silver plate coated with an emulsion of eosine and photographic gelatine was immersed for a few minutes in a strong solution of alum.

Naphthalene red, a substance soluble in alcohol and only slightly soluble in water, was also used in place of eosine. A little of this was dissolved in alcohol and then emulsified with gelatine. It was then poured on a silver plate, and the film was allowed to set thoroughly. The photoelectric currents produced were not quite so strong as those obtained with eosine. The effect is much greater in the blue than in any other part of the spectrum, the exposed plate being positive; and, apparently, with strong red rays, there is a reversal of the sign of the E.M.F.

Various other substances were used, among which I may mention iodine green, which gives rather strong currents with daylight, the exposed plate being negative.

In all these cells the E.M.F. is feeble compared with the electromotive forces which I have obtained by other means. to be presently described; but in the case of silver plates coated with iodine green, an E.M.F. of about $\frac{1}{20}$ volt can be observed with sunlight.

Fluorescine used like eosine on silver or platinum plates gives poor results.

A few experiments with a solution of sulphate of quinine and some other fluorescent liquids were made with a view to connecting fluorescence with electric effect; but the observed results were comparatively small, and the subject has been left for further trial.

Other Metals.

Many experiments were made with other metals, but the currents produced by most of them were smaller than those given by silver plates. Thus, I have on record an experiment in which bismuth was melted into thin plates which were fixed on glass, and then immersed in a cell containing water. When light fell on one of these plates, a considerable current was generated, the exposed plate being positive; but on adding a solution of chloride of bismuth to the water, the current generated by light was reversed. On taking them out of the cell and washing them with distilled water, and then immersing them again in a water-cell, when the plate was exposed to light, there was an impulsive current corresponding to a negative nature of the plate, but this was promptly succeeded by a very large current in the opposite direction—just as in the cases previously mentioned.

Copper, antimony, and other metals were also tried, but the results are not of sufficient importance to demand any detailed record. In particular, iron plates coated with the magnetic oxide and immersed in water yielded photoelectric currents.

Tin Plates.

A glass cell containing a solution of fluorescine in distilled water, in which two silver plates had been immersed, was taken for trial with other than silver plates. Almost the first metal that I used in this cell was tin. Two plates of tin foil whose surface had not been in any way prepared, but which was in the state in which it was obtained from the manufacturers, were fixed on plates of glass and placed in the cell. One of them being screened from light and the other exposed, a very strong current was the result. When the tin plates were replaced by copper, a current, but of less strength, was also produced. Gold plates gave no current at all. It was found, however, that the fluorescine was unnecessary, and that when common tap-water was used in the cell, the currents produced by light seemed to be no weaker than before. If the hand or any other screen were moved rapidly in front of the exposed plate, the spot on the galvanometer-scale moved correspondingly. A telephone was put into the circuit with a view to the production of sound by variations in the incident light, but the alterations were not sufficiently rapid to produce this result. A battery of three cells in series was then formed, but no sound was produced in the telephone, and, moreover, the current indicated by the galvanometer was no greater than when only one cell was used. This latter fact seemed most extraordinary; but, on examining the cells separately, I found that one of the cells was producing a current in a sense opposed to that of the current given by the other two. Of this more presently.

When the liquid in the cell was distilled water, currents were still produced by the action of light on the tin. When a drop of sulphuric acid was added to the water, the currents ceased.

In nearly every cell that I used with tin plates—whose surfaces, as stated above, had not been treated in any way—the exposed plate was *positive* to the unexposed; but, after a time varying from a few minutes to a few hours, it was found that this positive current died out and was replaced by an apparently stronger current, in which the exposed plate was *negative*. Thus there was a change in the sign of the E.M.F. produced by the continuous action of light. This again reminds us of M. Becquerel's observation about the thickness of sensitive layers. There seemed to be almost no exception to the rule that the exposed plate begins by being positive and ends by being negative, the negative regime lasting for many days of prolonged exposure to light.

It was also found that when the chloride of any substance was dissolved in the distilled water of the cell, all photoelectric action ceased. Again, if the tin foil is cleaned by immersion in sodic hydrate and then in hydrochloric acid, or by immersion in a solution of either of the chlorides of tin, the plate becomes completely insensitive to light. Thus the photoelectric result is obviously due to some layer on the surface.

In connexion with the change of sign of the E.M.F.

produced by continuous exposure, the following remarkable experiment was made. Round the outside of a cylindrical porous pot was fixed a coating of tin foil; inside the pot was placed a strip of tin foil, these two strips being the exposed and unexposed plates of the cell respectively. The porous pot was filled with water and immersed in a glass beaker also containing water, the diameter of the beaker being very slightly greater than that of the porous pot. The strip of foil inside the pot was completely covered from light and connected with one terminal of a galvanometer; the foil outside the pot being connected with the other terminal. Outside the glass beaker, and fitting round it very closely, was a cylinder of black paper with a vertical slit cut in it, the breadth of the slit being about half an inch, while the diameter of the cylinder was about 3 inches. Thus, by rotating the black paper cylinder round the beaker, different strips of the tin foil on the outer surface of the porous pot could be successively exposed to light.

The slit in the paper occupying a given position, the corresponding portion of the tin foil was exposed to sunlight. The current indicated that the plate was positive, and the exposure was continued until the current changed its direction, *i. e.*, the plate became negative. The slit was then moved opposite another and distant portion of the tin foil, which, by the same process, was finally rendered negative; and so on all the way round. Thus the surface of the tin foil was divided into a number of strips, which were alternately negative and positive in their electromotive forces when exposed to light; and by rotating the black paper continuously round the beaker, a series of currents in contrary directions were obtained from the action of light on one and the same metallic plate—a result which, at first sight, sounds very strange. It is easily understood, however, when we remember the different conditions into which the various strips of the surface of the plate—*i. e.*, some very thin stratum on the tin—were put in the preliminary process; and, moreover, it prepares us for cases in which different portions of a tin plate which has been sensitized by a special process give, on exposure, electromotive forces of different signs.

I have not yet spoken of any process for producing a sensitive layer on the surface, and a long time elapsed before I discovered one. The tin foil hitherto spoken of is pure tin foil as it is obtained from the maker; and, as must be expected, not only were some pieces on exposure to light observed to be at once negative while others (the great majority) were positive, but there were various degrees of sensitiveness. Occasionally, also, the curious jerks immediately on exposure and immediately on screening, which were observed in the case of silver and eosine, were observed with these plates.

Then succeeded a series of experiments in which a quadrant electrometer was used instead of the galvanometer, *i. e.*, the E.M.F., and not the current, was measured.

A curious result was very soon observed with the thin tin foil which is wrapped inside small packages of tobacco. The two sides of this foil are notably different in appearance, one being somewhat dull, while the other has a bright silvery aspect. Two strips of this were cemented on opposite sides of a small glass plate, the dull surface of one and the bright surface of the other being fixed to the glass. This plate was immersed in a glass cell containing absolute alcohol, the two pieces of tin foil being connected with the poles of the electrometer. When the bright surface was exposed to daylight, a small deflexion was obtained which showed the plate to be negative; when the dull surface was exposed, the deflexion was more than doubled, the plate being, like the other, also negative, and the magnitude of its E.M.F. being about $\frac{1}{18}$ volt.

A tin-foil plate, which at once on exposure to light is negative, is produced by thoroughly cleaning a piece of foil and coating it with *sulphide* of tin, either by exposing it to SH_2 or by rubbing a little "mosaic gold" over the surface.

When distilled water was used instead of alcohol, the E.M.F. produced by light was in all cases diminished; *and the same result always happened when any salt whatever was dissolved in the alcohol with a view to diminishing its resistance.*

This unfortunate result is characteristic of every photoelectric cell that I have employed. It is unfortunate, because it seems to preclude the possibility of obtaining really strong and practically useful *currents* by the action of light, and it appears to indicate some essential connexion of *resistance* and *electromotive force.*

The liquid is, of course, an essential element of the cell, and a very large number of liquids was tried in these experiments. Thus, for example, the two tobacco-foil plates when immersed in peroxide of hydrogen were absolutely insensitive. The alcohols are much the best liquids when tin plates are used. Large electromotive forces have been obtained with all of them; but they are not all of quite equal value. As has been already stated, if any substance containing chlorine is put into the liquid, the E.M.F. of light completely ceases.

If a small quantity of nitrate of potassium is dissolved in the alcohol, a Thomson high-resistance galvanometer will be very easily and strongly affected by the photoelectric currents of these cells.

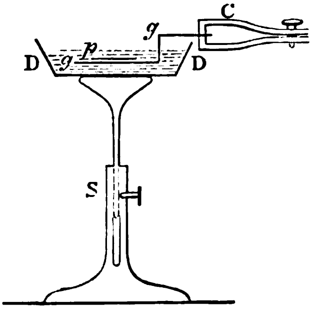
Among liquids an aqueous solution of pyrogallic acid is one which presents itself as worthy of trial, since pyrogallic acid is a greedy absorber of oxygen ; and I have on record a striking result of its employment. A small plate of tin foil, the surface of which had been cleaned, was covered over with a thin layer of water, with which a drop of nitric acid had been mixed ; the foil was placed on a glass plate, under which was applied a gas-flame until the liquid had quite evaporated, and the tin foil remained coated with some salt of tin. Whatever this salt was, when this plate was immersed in a cell containing absolute alcohol in presence of a cleaned tin-foil plate, the first was, on exposure to light, strongly *negative* to the second. The sensitive plate was then taken out of this cell and immersed in another containing a clear aqueous solution of pyrogallic acid. It was now absolutely insensitive to light. Taken out of this cell, and dried with blotting-paper, it was again put into the alcohol ; and now, on exposure to light, it was found to be strongly *positive.* Here, then, is a case of complete reversal. This change was accompanied by a slight jerk or kick in the E.M.F., such as has been already described in connexion with silver and eosine ; that is, on exposing the plate for the second time in alcohol, the initial indication of the electrometer was a very slight kick indicating a negative E.M.F., but this was immediately followed by a large deflexion in the opposite direction. This result was repeated many times; the plate was taken out of the alcohol and dried in the dark for about two hours, and when replaced in the alcohol cell the jerk and large contrary deflexion were obtained.

A similar experiment has been recently tried with a tin-foil plate sensitized by the process which I now adopt, and which will be presently described, and the result obtained was not the same as the above. In the new process the plate *appears* to be coated with the white oxide of tin, SnO_2; but whether it really is so or not, I cannot say. However, such a plate which had been in a cell with methyl alcohol for two years was taken out, together with the back or unexposed plate, and immersed in a cell with an aqueous solution of pyrogallic acid. On exposure to light in this cell, the plate was about half as sensitive, and of the same sign (positive) as in the methyl alcohol; and on replacing it in the alcohol cell, its indication was less than it had been originally. Thus, there was merely a falling off, but not an absolute destruction, of the E.M.F. caused by the pyrogallic acid, the sensitive surface having assumed a yellowish appearance.

The conjecture of M. Becquerel, that the sign of the E.M.F. developed by light depends on the *thickness* of the sensitive layer on the plate, has been already mentioned; but it may be allowable to put forward another conjecture—that when there are two, three, or any number of salts of a metal, any one of which may be on the plate, the result depends on the particular salt, and that the series may be alternately positive and negative on exposure to light according to the amount of oxygen (suppose) that they contain. A reducing agent may convert one of these into an adjacent one, and so alter the sign of the E.M.F. The question is not one with which I am competent to deal; it is for the chemist. And it might also be worth while to investigate whether there is any connexion between the sign and magnitude of the E.M.F. generated by the action of light on a metallic salt and the effect produced on the salt by electrolysis.

Production of the Sensitive Surface.—After trying a very great number of processes for producing a sensitive tin surface which is *positive* on exposure to light, the following is that which I have found to be most efficacious.

Fig. 2.



Take a strip of *pure* tin foil, say 1 inch long and $\frac{1}{8}$ inch broad, and about as thick as a stiff sheet of note-paper (the very thin foil is not suitable ; but the exact thickness is not of much consequence) : clean it first in sodic hydrate, and then in dilute hydrochloric acid, or in hydrofluoric acid, until the surface presents a bright crystalline appearance. Having thoroughly washed it in distilled water until every trace of acid disappears from the water, place it horizontally, at *p*, on a bent plate of glass of the shape *g g* in fig. 2, this plate of glass being held by a clamp C supported by an ordinary retort-stand, which is not represented in the figure ; under the tin plate *p* and the glass place a dish, D D, supported by a stand S, which allows the dish to be raised or lowered ; the dish being raised until it nearly touches the glass *g*, pour into it, until the tin plate *p* is completely covered, a portion of the following mixture :—

500 c. c. of distilled water,
3 c. c. of pure nitric acid,
15 grammes of nitrate of ammonia.

If the tin *is pure* and has been *thoroughly cleaned*, the moment the plate *p* is covered by the solution, a whitish deposit is thrown down on its surface. The plate may be left in the solution for 3 or 4 minutes, and then the dish D D is lowered and removed. The deposit on the tin should be uniform and must not be allowed to form for more than 4 minutes. When the dish D D has been removed, the under surface of the glass plate *g g* should be dried by blotting-paper. Then a process of heating must take place. The flame of a spirit-lamp must be applied uniformly to the under surface of the glass plate, *i. e.* moved backwards and forwards until the whole of the liquid on its upper surface is evaporated. Great care must be taken not to melt the tin plate *p*.

Continuing the heating process, the upper surface of the tin passes through a series of appearances. At first (*i. e.*, when the water has just been evaporated) the deposit has a dull slaty look; as the heating goes on, this changes to a whitish aspect, a gas with a nitrous smell coming off; as the heating is continued, this whitish surface undergoes a rapid change to a dark colour with, apparently, a tinge of green, which travels like a shadow from one end of the plate to the other; the heating being still continued, this dark surface changes to a strong white, and the flame ought to be kept under it until the gas is completely driven off.

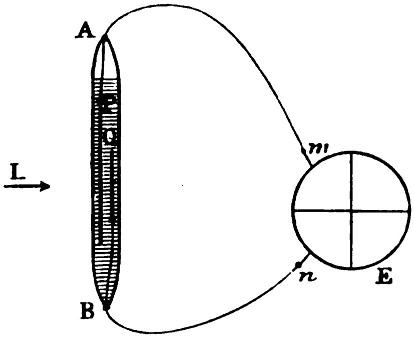
The plate should then be plunged into alcohol. It is now in the most sensitive condition. I have found that if the heating is stopped at the end of the penultimate stage—viz., that in which the dark greenish colour has been reached—the plate will be very fairly sensitive to light. It may be suspended from a platinum wire fixed through a pin-hole at one end of the plate, and when placed in an alcohol cell in front of a clean tin plate similarly suspended, these plates being connected with the poles of an electrometer while the cell is screened from light, a small difference of potential will be observed, the sensitized plate being positive to the unsensitized; but this difference of potential will usually disappear after a short time.

Many liquids can be used in the cell; but I have found that the best results are obtained from *methyl alcohol prepared from oil of wintergreen*. With methyl alcohol prepared from wood-spirit I failed to obtain anything like the maximum E.M.F. on exposure to light. A plate so prepared is, on exposure to light, positive to the clean plate in the cell. In some liquids the plate is almost quite insensitive to light, but on replacing it in methyl alcohol its sensitiveness reappears.

It is remarkable that the plate while immersed in the cell (and unexposed) takes a considerable time—about five hours—to develop its maximum sensitiveness; that is, the cell should be left in the dark for this time to allow the plate to develop. Moreover, shortly after the plate has been formed, it is much more rapid in its response to light and shade than it is subsequently; not that there is any falling off in the final indications of E.M.F., but that the indications are more slowly produced.

Fig. 3 represents a cell connected with the electrometer. AB is a small glass tube nearly full of the liquid; P and Q are the sensitized and unsensitized plates, two fine platinum wires being either soldered to them or passed through minute holes in them, these wires being sealed into the glass tube at A and B, and connected with the poles *m*, *n*, of a quadrant electrometer, E; the arrow L represents the incident light.

Fig. 3.



After forming the sensitive plate, it should be tried in the cell before the cell is sealed up completely; if it is not satisfactorily sensitive, it should be taken out and re-heated either on the glass plate in fig. 2 or over the chimney of a paraffin lamp. This further heating will often convert a partially sensitive plate into a very sensitive one.

The cell is usually fixed in a piece of cork by means of which it may be held in a support.

Dispersion of the Residual Effect.—On the withdrawal of the light, the fall of E.M.F. in the cell is usually much slower than the rise of E.M.F. on exposure; and this fact would constitute a grave inconvenience if there were no speedy remedy. The effect of the light can, however, be quickly and satisfactorily overcome by connecting the exposed plate with the copper, and the unexposed with the zinc pole of a Daniell cell for a few seconds—the time of connexion being longer as the time of exposure of the plate to light was longer. In fact, a series of three or four impulsive contacts with the poles of the Daniell, followed by a few seconds' short-circuiting, will suffice to remove the residual effect of light, and to leave the spot on the electrometer-scale at the point from which it started.

This result is important, because when feeble light, such as that of a candle, falls on the cell, the maximum E.M.F. takes some minutes to develop, and the return of the spot on the scale would occupy a long time.

Variation of the Effect with the Distance of the Source of Light.—Six cells connected in series were placed on circles of varying radius, and a candle was in each case placed at the centre of the circle. The E.M.F. developed by the light of the candle was, with fair accuracy, found to be inversely proportional to the distance of the candle from the cells. As the intensity of the light varies inversely as the square of the distance, it follows that the square of the electromotive force is proportional to the intensity of the light.

Curve of Rise of E.M.F.—The law of increase of E.M.F. during exposure was studied by placing a "standard" candle at a distance of 6 inches from 6 cells connected in series, the poles of the series being connected with a Thomson quadrant-electrometer. The deflexions on the scale were noted every quarter minute, and a curve was traced having the deflexions for ordinates and the times for abscissæ. The maximum E.M.F. attained was ·566 of that of a Minotto cell, giving for each cell ·094 of this amount.

If we denote by A the maximum E.M.F. developed, and by η the E.M.F. at any time t, it would appear to be legitimate to assume the equation

$$\frac{d\eta}{dt}=k(\mathrm{A}-\eta),$$

where k is a constant. This gives, by integration,

$$\eta=\mathrm{A}+\mathrm{B}e^{-kt},$$

where B is another constant. The curves actually traced in the experiments closely satisfy an equation of this form, which is, of course, that of a logarithmic curve.

The curve of fall is probably of the same nature, but the rate of fall is much slower than that of rise, as is evident by an inspection of the diagram exhibited to the Society.

Unhomogeneous Surfaces.—To produce a homogeneous surface on the sensitive plate, it is manifestly necessary to satisfy two conditions: the deposit produced by the oxidizing solution must be uniform, and so must the heating by the spirit-lamp. It is obvious, then, that if these conditions are not fulfilled, we must be prepared to find some part of the plate positive and another part negative on exposure to light; and such has actually been found to be the case in many experiments. But a much more extraordinary result of quite common occurrence is that described under the next head.

Impulsion-Cells.—It often happens that, a few days after a cell is mounted and found to be duly sensitive to light, an exposure produces no E.M.F. at all. But if a slight tap (sometimes scarcely audible) is given either to the support of the cell or to the table on which this support rests, a change, indicated by the motion of the spot on the electrometer-scale, takes place in the cell, and it is as sensitive to light as it was originally. Another tap given to the base throws the cell again into the insensitive state; another tap will restore the sensitive state; and so on indefinitely.

These results are not due to any defect in the contact of the platinum wires with the plates; these wires are tightly pinched to clean parts of the plates, and often soldered to them.

It might be conjectured that these results are due to the formation of some gas in the cell; but how can one tap replace the gas which has been displaced by another tap?

I believe the sensitive and insensitive states to be due to some molecular alteration, either in the sensitive surface or in the liquid, or in their layer of contact—a sort of polarity in the medium which, as in the case of magnetic bodies, can be produced or destroyed by vibrations. Thus the E.M.F. due to light may be simply due to strain and not accompanied by any actual chemical combination.

A most remarkable instance of these impulsion-effects may be cited to show that they are not due to any defective contact. Referring to fig. 3, which represents the sensitive plate completely submerged in the liquid, let the plate be only partially submerged, and let the liquid surface be at a point C, about midway between B and A, the portion CA of the cell being occupied by vapour alone. This will be the state of affairs in a cell which was under experiment. It was found that on exposing the upper portion, CP, of the plate and screening the lower, a *negative* deflexion on the scale was observed—that is, a deflexion which indicated that the exposed plate was negative; while if the upper portion was screened and the lower exposed, a *positive* deflexion resulted. Then on giving a slight tap to the support of the cell and exposing the portion CP, while screening the lower, a *positive* deflexion was observed; and on exposing the lower portion alone, a *positive* E.M.F. resulted, as before. Thus the effect of the vibration was to alter the nature of the upper portion while leaving the lower unaltered.

In another cell, which I now exhibit to the Society, the sensitive plate was completely immersed, and there was, as in the last case, a certain portion, CP, at the upper end which was rendered alternately positive and negative by vibrations, while the lower portion remained unaltered. These facts seem to be quite inconsistent with a want-of-contact theory. Before citing another experiment, in which a Thomson galvanometer is employed, to disprove such a theory, it is well to mention a remarkable method of producing the sensitive state from the insensitive.

While investigating the effect of static charges communicated to the plates on the sensitive and insensitive states, I found that if a Voss machine, not in any way connected with the cell or the electrometer, was worked in the room while the cell was in the insensitive state, *the moment a spark passed between the poles of the Voss, the insensitive state was altered to the sensitive*, whether the cell was connected with the electrometer or not. The same effect was produced at a much greater distance from the cell by the inductive action of the spark passing between the two poles of the secondary coil of an induction-coil, as in a Hertz oscillator arrangement. In fact when a Hertz oscillator was taken into the grounds outside the laboratory in which I worked, the induction-coil being actuated by a battery of 4 or 5 Grove cells, no wires whatever passing from this machine near the laboratory, the insensitive cell inside the room at a distance of 81 feet was instantly rendered sensitive by the inductive action of the Hertz arrangement.

Impulsion-cells differ much from each other in the readiness with which they undergo the change from one state to the opposite; but the most obstinate can be thrown from the insensitive into the sensitive state by leading a wire connected with either pole into the vicinity of a Hertz or a Voss machine.

I am not able to produce the reverse effect—the change from the sensitive to the insensitive state—by electromagnetic induction; dull taps administered to the base of the cell constitute the only way in which this change can be produced with certainty. On very rare occasions the change has been effected by the inductive action of strong sparks from a Leyden jar; but the result is quite exceptional. There appears to be some reason for supposing that this change—from sensitive to insensitive—is produced by vibrations of slow period, or very dull taps. I have produced it by dropping very small pieces of cork on the base of the iron retort-stand in which the cell is held, and even by gently drawing a piece of paper across the retort-stand.

Nearly all these tin-foil cells will develop the impulsion character a few days, or weeks, after they have been formed, *provided that the alcohol in them has not been thrown out and replaced by fresh alcohol.* Three years ago I formed a battery of 30 of these little cells, testing each before adopting it in the battery, and connected them in series, hoping thereby to obtain a very large E.M.F. on exposure to light. To my disappointment, however, I found a comparatively small result; and on examining the cells individually, I found some of them insensitive. Their sensitiveness was restored by renewing the alcohol, but it might have been restored by impulses if I had known the fact.

One of this battery in which the impulsion results appeared was taken out and its alcohol renewed on 4 or 5 days successively, with a view to ascertaining whether the impulsion results could be got rid of by renewing the liquid. After six renewals the cell ceased to give impulsion-effects, and it has been since under trial from time to time for about three years.

Impulsion-cells are very strongly influenced if they are kept in a room in which powerful sparks are being produced from a Holtz machine. A box of them which had been put by, for exhibition before this Society, in a cupboard about six feet distant from a Holtz which was put in action, was found during the process to have had nearly every cell affected in such a manner that no impulsion-effects could be produced for several hours after the Holtz ceased to be worked ; and a strikingly good example of an impulsion-cell which I was very anxious to preserve acquired a strong tendency to revert from the sensitive to the insensitive state, and to remain in the latter ; at the same time it lost its extreme sensitiveness to impulses.

The experiment with a galvanometer, previously mentioned, to show that in neither state of an impulsion-cell is there a want of contact, is as follows:—An impulsion-cell in which both plates hang from the top of the cell, and in which the liquid did not rise nearly to the level of the junction of either platinum wire with the plate, was made part of the circuit of a galvanometer and a Daniell cell by means of a key ; its poles were connected with the electrometer, and before pressing down the key which put the cell into the galvanometer circuit, it was made insensitive by an impulse. On pressing down the key, the needle of the galvanometer was deflected, and this could not have happened if either contact were broken. In fact, the indication of the galvanometer was the same whether the cell was in the sensitive or in the insensitive state.

In two or three cells in which the alcohol had partially evaporated, leaving a portion of the sensitive plate above the liquid, it was found that, the whole plate having been originally positive, the portion in the vapour gave a negative E.M.F., while the lower portion continued positive.

Among the numerous liquids tried in these cells was butyric acid, which is efficacious ; but after a few days it acts on the tin plate and tends to destroy the sensitive surface. A plate which had been kept in a cell containing butyric acid for some days was removed into one containing propylic alcohol. The effect was that the upper portion of the plate exhibited impulsion-effects—being alternately positive and negative with impulses—while the lower portion remained positive.

In acetone, nitrate of amyl, and glycerine these plates are also sensitive—as indeed in water also ; but the E.M.F. in these liquids is less than in methyl alcohol.

It was found also that if the cell contained only the vapour of alcohol, an E.M.F. was generated by light, but less than that in the liquid. In aldehyde no E.M.F. due to light was observed.

Hydroxyl has the effect, after about twenty-four hours, of giving to the plate a yellowish colour, and also of changing the sign of the E.M.F. which exists in the dark.

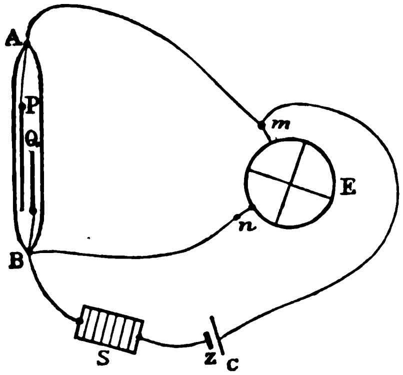
It has been found many times that one effect of removing a sensitive plate from an alcohol cell to a cell filled with hydroxyl or other liquids and then replacing it in the alcohol, was to develop impulsion results ; and this fact shows that these results cannot be due to any induced electrification on the glass of the cell and an action of light on this electrification —such an action as has been the subject of recent experiment by a Russian physicist.

If, then, I am right in supposing that the development of electrical charges on these sensitive plates is in some way connected with molecular disturbances produced by electromagnetic induction or mechanical vibration, it must be admitted that the result is possibly a very important one, and that it may play a large part in the economy of nature. Thus, the mode in which solar energy is taken up in the cells of plants may be largely dependent on such disturbances in the atmosphere or in the earth. And in this connexion a somewhat wild conjecture may be pardoned.

The microradiometer of Prof. Boys works wonders in the measurement of very minute changes of temperature. If, then, we imagine a beam of light to be incident on the sensitized plate of an impulsion-cell from which it is, in part, reflected to a microradiometer, when the cell is in its insensitive state the incident energy is not taken up electrically and statically, and a greater portion of it would be sent to the radiometer than would be sent if the cell were in the sensitive state, so that a mechanical tap or an electromagnetic impulse, in altering the cell from electrical insensitiveness to sensitiveness, should produce an effect on the radiometer.

Connexion with the Daniell Cell.—It has been already stated that the residual effect when light is withdrawn from the cell is got rid of by connecting the cell with a Daniell cell in such a way that the deflexion caused by the latter is opposed to that produced by light—*i. e.*, the sensitive plate, P (fig. 4), is connected with the copper pole of the Daniell.

Fig. 4.



A very curious result of this arrangement deserves to be mentioned, inasmuch as it may possibly afford a clue to the nature of the action of light on the sensitive plate. Between the Zn pole of the Daniell and the plate Q interpose a very great resistance, S, of the same order of magnitude as the resistance, R, of the photoelectric cell itself—*i. e.*, several megohms. This resistance S is composed of lead lines traced carefully on glass and then covered with shellac, and is, in my experiments, something like 10 megohms. The poles, A, B, of the cell being connected with the electrometer, if e is the (disturbing) E.M.F. of the cell in the dark (which may be zero or very small), and E that of the Daniell, we shall obtain a deflexion, Δ, given by the expression

$$\Delta = \pm e + \frac{R}{R+S}(E \mp e),$$

the signs $\pm$ being taken according as e produces a deflexion in the same sense as E or in the opposite sense. This is on the assumption that e is not modified by E, which is possibly false, but not material to the result. Whether e is or is not modified by E, it is clear that if S is very small compared with R, the deflexion on the scale will simply indicate E, no matter how great e may be; and hence if light is allowed to fall on the cell with this arrangement, *there will be no indication of its effect on the scale.* But taking S of the same order of magnitude as R, we obtain, when the cell is in the dark, a deflexion of, say, half the amount produced by the Daniell alone. When the connexions are those indicated in fig. 4, *i. e.* when the Cu pole is connected with the sensitive plate, if light is allowed to fall on the cell a very large deflexion of the spot (of course in the direction opposed to Δ) is produced. If after this we reverse the connexions, *i. e.* connect the Zn pole with the sensitive plate, and allow the spot to settle to its position of rest in the dark, and then let the light fall on the cell, the deflexion produced by light is *very much smaller than before.* To quote a particular case—when the Cu pole was connected with the sensitive plate and the spot came to rest, the spot was deflected from this point through 260 divisions on the scale; and when the connexions were reversed and the spot again allowed to come to rest, it was deflected from the point of rest through only 50 divisions.

From the above expression it is obvious that, in the first mode of connexion with the Daniell, the deflexion from the zero produced by light is $e-\lambda+\frac{R}{R+S}(E-e+\lambda)$, where λ is the E.M.F. due to light, on the supposition that R is constant, so that the observed deflexion on the scale from the point of rest due to the Daniell is

$$\lambda\left(1-\frac{R}{R+S}\right), \text{ or } \frac{S}{R+S}\lambda;$$

and if λ were the same in the second mode of connexion, the deflexion would be also of this value; but since the deflexion is very notably less in the second case, it is almost certain that λ is also less.

Observe that in the first mode of connexion the action, or possibly only the tendency, of the Daniell is to deposit a layer of oxygen on the sensitive plate ; and if light tends to reduce oxygen at the surface, its action would be increased owing to the additional layer due to the Daniell ; while with the reverse connexions, the reducing action of light would obviously be lessened.

The electrolytic action of a Daniell battery on one of these cells is not permanent ; it disappears rapidly, and, indeed, it is not certain that anything more than a strain tending to electrolysis is produced.

The current of a Daniell cell, or any battery, if passed through an impulsion-cell, does not influence the state of the cell—*i. e.*, this current does not alter the sensitive to the insensitive state, or *vice versâ.*

Supposing S, and therefore Δ, to be such that when (in the first case) light falls on the cell, the spot on the scale is brought back just to the zero from which it started, if the Daniell is suddenly removed the light will then cause a further deflexion—which is the normal amount due to the light. This fact agrees perfectly with the theory of a reducing action, because, in the position of equilibrium of the spot when both light and the opposing Daniell act, the rate at which light tends to reduce is equal to that at which the Daniell tends to deposit oxygen ; and therefore in this particular case the plate is really in its normal state, so that when the Daniell is removed, light finds the plate in the condition in which it would be if no Daniell cell were connected with the photoelectric cell.

Action of Different Colours.—In a sensitive tin-foil cell, the action of the blue part of the spectrum is very much greater than that of any other part ; but measurable results can be obtained all through, as will be seen by the diagram exhibited to the Society. The spectrum was that of lime-light passed through a prism of bisulphide of carbon, a single cell being used in the experiment. This cell has been, from time to time, under experiment for more than four years, and its action is now exhibited to the Society. It has not appreciably deteriorated, but the time of any one exposure has never been more than a few minutes.

Seleno-Aluminium Cells.

In the year 1880 Mr. A. Graham Bell utilized the property of conductivity possessed by a modification of selenium which had been previously found to be a conductor, and also the property of variable conductivity when light of variable intensity falls upon it. These properties, as is well known, were utilized in the reproduction of sound by means of a telephone and a battery in the external circuit of which was placed a sensitive selenium conductor. This selenium arrangement is usually called a "selenium cell," but a *selenium conductor* or a *selenium resistance* is a much more appropriate term.

When the photophone was announced, selenium resistances were made in this country first, I believe, by Mr. Shelford Bidwell, who showed some of them at the last meeting of this Society, and by means of them produced two results of striking beauty, which I shall presently endeavour to reproduce by different means.

Last year I set about constructing selenium cells, properly so called—that is, cells in which electromotive force is produced by the action of light. The method adopted was to take two small clean plates of any metal, to spread a thin layer of the already recognized sensitive selenium on the surface of one of them, and, connecting each with a fine platinum wire, to immerse them in presence of each other in a small glass cell containing a liquid. Thus a large number of metals and a large number of liquids had to be tried for the best result. Plates of platinum, silver, tin, copper, zinc, bismuth, mica, glass, and other substances were tried with various liquids. With copper the E.M.F. produced by light was almost, if not quite, zero. All the others gave considerable results; but much the best result was obtained with plates of aluminium; and for some time the liquid used was one of the alcohols—preferably methylic. In the course of a few days, however, the aluminium plates in alcohol were found to be covered with a kind of gelatinous deposit, which, I am told, is an aluminate of alcohol. This liquid was, therefore, abandoned; and the best result of all was found to be produced with *acetone.*

The process of forming the sensitive plate is as follows:—On an iron tripod is supported a porcelain plate which is heated from below by a Bunsen flame; the little strip of cleaned aluminium is placed on this plate, and when it has got hot, one end of it is held in a forceps, while a drop of melted selenium placed at the end of a very hot glass rod is rapidly smeared over the aluminium plate. This selenium layer should be of uniform thickness, and the thickness must be neither very great nor very small. When the layer is deposited, the aluminium plate is quickly removed from the porcelain plate by the forceps and rapidly moved up and down in the air for a few seconds, the gas-flame being, at the same time, removed from under the porcelain plate. Both plates having now become slightly cooler, the aluminium plate is replaced on a comparatively cool part of the porcelain plate, and any tendency of the selenium to become liquid is checked by blowing over its surface. Working the gas-flame now rapidly backwards and forwards under the porcelain plate and occasionally blowing over the selenium surface, a series of changes—very much resembling those presented in the preparation of a sensitive tin surface—are observed. The appearances are as follows:—

1. The originally jet-black selenium surface gradually assumes a bluish-white appearance.

2. As the process is continued, this latter surface becomes a grey which may be of several shades. It may be a light grey, or a grey with a violet tinge, or a grey with some glossy spots or streaks. None of these surfaces is to be accepted. They are the final forms which most readily present themselves; and when they do, the selenium must be melted afresh and again spread over the aluminium plate—the whole process being repeated with its gradual heatings and coolings, until finally—

3. The surface of the selenium assumes a very dark brown colour.

This is the most sensitive surface that can be obtained. At first, accepting too literally the statement that "the grey modification of selenium is the sensitive one," I accepted every plate which finally assumed a grey appearance, and constructed a large number of cells for a battery. By accident, however, a plate with the brownish colour was formed, and it proved to be so much superior to the others, that they were all rejected.

The glossy spots and streaks which sometimes exist on the grey surfaces are, I think, due to an indefinitely thin layer of the black selenium which has escaped the necessary transformation; and to observe them, it is well to look at the plate almost in the plane of its surface. The dark brown surface is devoid of them, and is in appearance quite homogeneous.

When the plate has assumed this appearance, it may be screened from light and left on the porcelain plate to get cool, for which about ten minutes will suffice. When the plate has cooled, it can, apparently, be kept in the dark, unimmersed in any liquid, for any length of time before being put into the acetone cell. A plate was thus kept for sixteen days, and then, on being placed in the acetone cell, it was as sensitive as if it had been immersed immediately after formation. It is a marked peculiarity of the seleno-aluminium cell that, immediately after it has been set up, it is wonderfully rapid in its response to light, and that on the withdrawal of the light the E.M.F. at once disappears; but after a few days it is much slower in both respects—particularly the latter—while its sensitiveness as regards the magnitude of the E.M.F. developed is unimpaired.

The dispersion of the residual effect is produced by the means before described for the tin cells, viz., connexion with a Daniell cell, the sensitive plate being now, of course, connected with the Zn pole of the Daniell.

No sensitive and insensitive states due to vibrations, mechanical or electromagnetic, have, so far, been observed in the seleno-aluminium cells.

Sign of the E.M.F. due to Light.—Unlike the tin-foil plates described, the sensitive plate in a seleno-aluminium cell is strongly *negative* towards the insensitive plate when the cell is exposed to light.

Effects of Different Colours.—The seleno-aluminium cells differ from all other photoelectric cells that I have constructed in their great sensitiveness to all parts of the spectrum, the maximum effect being produced in the yellow near the borders of the green.

No very accurate experiment has yet been made on this subject, because the Thomson quadrant-electrometer at my disposal happens to be out of order; but with Clifton's form of the instrument, in which great sensitiveness has been aimed at rather than accuracy or constancy, the following numbers represent the relative electromotive forces produced by the spectrum of an albocarbon light formed by a bisulphide-of-carbon prism:—

Red	109
Border of red and yellow	117
Yellow	130
First edge of green	113
Last edge of green	101
Middle of blue	104
End of blue	102

The E.M.F. of a Daniell was represented by 408, and the cell was about one metre from the prism. Most probably, of course, the number 101, corresponding to the border of green and blue, is less than it should be.

All other photoelectric cells may almost be said to be sensitive to the blue alone.

Effect of Continuous Exposure to Light.—The effect of exposing a seleno-aluminium cell continuously to daylight depends on whether the cell is left on open or on closed circuit. A cell left on open circuit for many hours, and then kept in the dark until its E.M.F. settles down, will be found to have fallen to one fifth of its original value; but if then kept during the night and observed in the morning, the E.M.F. produced by exposure to light will have quite recovered its first magnitude. Not so with the cell on closed circuit; its E.M.F., on fresh exposure, will be reduced to nearly one half its original value.

A cell which had been exposed to daylight for five days (with, of course, the advantage of each night's resuscitation) is now shown to the Society, and its action can be compared with that of a fresh cell.

Connexion with a Daniell during Exposure.—When a Daniell with a very great resistance interposed, as described in connexion with the tin cells, is connected with a seleno-aluminium cell, so that the Zn pole is first connected with the sensitive plate, and then with the insensitive, the E.M.F. developed by light is much greater in the first case than in the second—a result which is the same as before, since the seleno-aluminium plate is the negative one in the cell, and therefore the effect of this connexion with the Daniell is to develop between the poles of the photo cell a difference of potential opposed in sign to that which light produces.

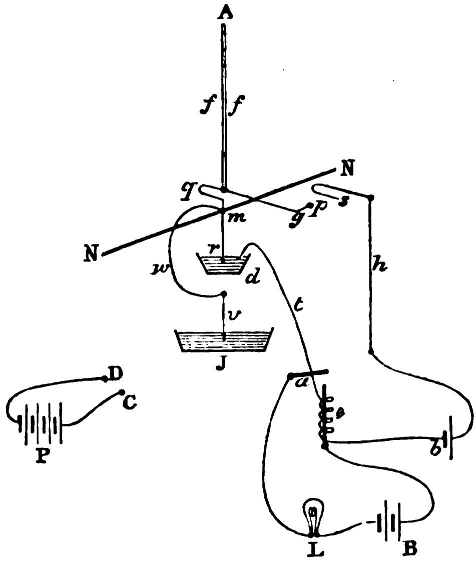
Mechanical Effects.—The E.M.F. generated by light in a photoelectric battery can be utilized for ringing electric bells, lighting or extinguishing electric lamps, and possibly other things, although the materials of the battery are never used up in producing currents. The method which I have employed consists in utilizing the motions of an electrometer needle (due to the E.M.F. of the photoelectric battery) for making a contact and completing the circuit of an ordinary voltaic battery, whose current, thus completed, rings a bell, lights a lamp, &c.

The first arrangement which I employed for this purpose was as follows:—At the middle of the aluminium needle of a quadrant-electrometer, and at right angles to the length of the needle, is fixed a very fine glass tube about 1¼ inch long; a platinum wire traverses the interior of this glass tube and, coming out at its ends, this wire terminates in two little spheres of platinum. Over the quadrants were fixed two small mercury-cups which were completely insulated from the quadrants and which were permanently connected with the poles of an external battery which had an electric bell in its circuit. When light shone on the photoelectric battery (whose poles were connected with those of the electrometer), the needle was deflected and, carrying the little glass tube with it, brought the platinum points into contact with the mercury in the cups; thus the circuit of the voltaic battery was completed, and the bell rung. This arrangement was unsatisfactory, owing to the difficulty with which the platinum points separated from the mercury, and it has been replaced by a much more satisfactory plan devised by Mr. Appleyard, to whom I beg to express my thanks for the untiring perseverance which he devoted to the perfecting of the apparatus.

The improved arrangement is as follows:—

In the figure N N represents the needle of the quadrant-electrometer, the quadrants of which are not represented;

Fig. 5.



g q m is a very fine glass tube passing down through the middle point, *m*, of the needle; a platinum wire, *p g q m r*, traverses this tube, the portion *g q* lying at right angles to the long diameter, N N, of the needle, and terminating in a sphere, *p*; the portion *m r* of the wire dips into a shallow dish, *d*, of mercury which is supported below the quadrants by an arm not represented; a platinum wire, *t*, dips into the dish *d*, and, passing outside the case of the electrometer, is connected with one end of the coil of a small electromagnet, *e*, whose armature is *a*; the other end of this coil is connected with one pole of a weak voltaic cell, *b*, whose other pole is connected with a small platinum plate, *s*, through a support, *h*, which is fixed inside the case of the electrometer, and, of course, insulated from the needle and the quadrants; the platinum plate *s* is fixed tolerably close to the position of rest of the platinum point *p*; and when the needle is deflected by the E.M.F. of a photoelectric battery, P, connected with the poles, C, D, of the electrometer, the point *p* comes into contact with the plate *s* and completes the circuit of the voltaic cell *b*, and thus brings the armature *a* into contact with the magnet of the coil *e*: this connexion completes the circuit of a voltaic battery, B, in whose circuit is an incandescent lamp, L, or an electric bell, which is then set in action. The needle is connected with the sulphuric acid jar, J, by means of a fine platinum wire, *w*, bent into a semicircular form so as to avoid contact with the dish *d* when the needle moves, the wire *w* terminating in a vertical length exactly under the centre of the needle; thus, since the wires *r* and *v* are in the vertical axis of rotation of the needle, no appreciable friction hinders the motion of the needle.

The needle is suspended by two fine silk fibres, *f*, *f*, from a support A inside the case of the electrometer. Instead of the *two* external voltaic batteries, B, *b*, *one* would theoretically suffice; but it is found that, to prevent sparking and "stiction" between *p* and *s* when the current is made, it is preferable to have a very weak current traversing this portion of the arrangement—one which is just sufficient to work the electromagnet *e*.

In this way, by means of a few seleno-aluminium cells, P, I have found no difficulty in ringing a bell by the light of a taper or that of a match held at a distance of a few feet from the battery P. Observe that by this method we never draw on the materials of the photo-battery, because no current ever passes through it; it is simply connected with the poles of the electrometer, and its E.M.F. alone is employed.

The Problems of Photoelectricity.

Three prominent problems in this subject deserve to be signalized. The first, and least pretentious, is the construction of a really scientific photometer. It cannot be said that the comparison of two lights by means of a spot of grease is a very satisfactory procedure; but in the seleno-aluminium cell we have, on account of its great sensitiveness and range in the spectrum, an approximation to the possibility of an electrophotometer—at least so far as the *comparison* of lights is concerned. For, assuming that, by means of such a cell, we take, as it were, each light to pieces, and note the intensities of the different colours—which are directly proportional to the squares of the corresponding electromotive forces—we could then apply the principle of Newton's chromatic circle to determine the value of the resultant light. The second problem—and that which in 1877 induced me to work at this subject—is the electrical transmission of an image to any distance; in other words, the construction of a telephotograph. The problem seems to be one of exceedingly great difficulty—much more difficult than the problem of the telephone—because the parts of an image are *simultaneous* and not, like the sounds of the voice, *successive*; and, indeed, we have also to deal with a quick succession if we are to transmit a living moving image, such as Homer depicted on the shield of Achilles. The early attempt which I made consisted in the construction of a cable, somewhat on the model of the optic nerve. The optic nerve consists of a bundle of fibres, each a conductor of electricity and each separated from its neighbour by being surrounded by a medium. One set of ends of this nerve abuts on the retina, which is its sensitized plate, and the other in the brain. It is well known that light incident on the eye causes a photographic decomposition on the retina; and I believe that images have been seen on the retina of a rabbit which was immediately killed after a strong light had been presented to its eye. The images, then, of external objects are transmitted along the optic-nerve cable to the brain, where by some means or other they result in a process of thought. Possibly thought is an equivalent of at least a part of the originally incident energy. No satisfactory solution of this problem will be attained by any slow and painful mechanical process of tracing out in succession the various portions of a picture; and it does not now seem that we are near any true solution of the problem, whatever startling stories the newspapers may from time to time report.

The third problem is the direct transformation of the radiant energy of the sun into work useful to us, without the consumption (at least on any large scale) of materials on the earth—in other words to get rid of that terrible waster of energy, the steam-engine. I usually find two objections made against any investigation in this direction. The first is, that the attempt is contrary to the principle of the conservation of energy. The objection, however, does not seem to be well founded; because it is conceivable that a photo-electric battery may yet be found which will simply act as a transformer of the energy which it receives from the sun, while its own materials, being merely the implements used in the process, may be almost wholly unmodified. The energy thus taken out of the sun may finally be radiated out into space from the earth in the form of heat, if it is true that all forms of energy must ultimately pass into this form —a proposition which, being a very wide generalization from our experience on the earth, it may be permitted to doubt in the universal necessity claimed for it.

The second objection is that there is not energy enough in the solar rays at the distance of the earth to supply the work desired. This objection is founded on the experiments of Pouillet, Violle, and others, who have estimated the solar energy incident per square foot per second on the surface of the earth. Let us see how much energy, according to the assumptions based on these experiments, is at our disposal.

The quantity of solar energy, assumed to be measured in metric thermal units, which is incident, normally, every minute on a square centimetre at the distance of the earth is given by the expression

$$A\,a^{\epsilon},$$

where, according to Pouillet, A (the solar constant) is 1·7633 calories, a (the atmospheric constant) is something between ·7244 and ·7888—let us say that $a = \cdot 75$; and ϵ is the ratio of the thickness of atmosphere traversed to the normal thickness of the atmosphere measured from the place of observation. M. Violle employs a formula of this form, but, according to him, ϵ involves the height of the barometer and the pressure of aqueous vapour present in the air, and $A = 2{\cdot}54$ calories. At the superior limit of the atmosphere ϵ is, of course, zero, and at the surface of the earth, for normal rays, $\epsilon = 1$. Taking this latter value of ϵ, and converting the thermal units into ergs, the quantity of energy incident per square centimetre per minute is

$$1{\cdot}3224 \times 42 \times 10^{6} \text{ ergs}$$

according to Pouillet, and

$$1{\cdot}905 \times 42 \times 10^6 \text{ ergs}$$

according to Violle. Converting these into foot-pounds' weight per square foot per second, we have

63·42 foot-pounds' weight per sq. foot per second (Pouillet),
91·35 „ „ „ „ (Violle).

The quantity of solar energy existing at any instant in one cubic foot of space at the surface of the earth is, of course, obtained by dividing these numbers by the velocity of the radiations in feet per second (roughly 10^9), and this quantity is infinitesimally small. Perhaps it is the infinitesimal value of the energy in a cubic foot which impresses people so strongly with the insufficiency of the solar radiation for doing work; but it is obvious that with this quantity we are not concerned. It is the amount which is contained at any instant in the immensely long column through which the energy travels in a second that is important.

The number assumed by Clerk Maxwell ('Electricity and Magnetism,' vol. ii. p. 402) is 83·4, which is very nearly what Pouillet's becomes at the superior limit of the atmosphere.

Thus we see that the amount of power at our disposal is small—yet, perhaps, not hopelessly small.

But is it necessary to accept this deduction from Pouillet's experiments at all? I think not, because it is not by any means certain that a blackened surface, such as that in Pouillet's pyrheliometer, catches up and transforms into heat every form of energy in the solar beam. There may be forms of energy which take no notice of blackened surfaces and which refuse to be converted into heat by means of them. Perhaps the proper receptive surfaces for them remain to be discovered; and it is this consideration chiefly which gives hope to the experimenter in Photoelectricity. The dissipation of every form of energy in the Universe into the final form of heat of uniform temperature is a dogma and nothing else—and, moreover, a dogma which leads to the most dismal results. Can its advocates trace the process in the case of energy incident on the retina, telephotographically transmitted to the brain, and converted into—or, at any rate, productive of—thought; and can they show that after this process nothing but heat remains?

It is not to the purpose to say that, "so far as we know," all forms of energy must run into heat finally, and be diffused throughout the Universe, because we may even still imagine that there are many very astonishing physical facts beyond what we know. Indeed, even in the unspeakably dismal event of the final conversion of the whole "visible" Universe (whatever that may mean) into one huge dark inert mass of uniform temperature in which no life is possible, this much, at least, is included in what we *do* know—viz., that we shall not have then got rid of the static energy of gravitation; and I strongly suspect that if we journeyed out to Antares or to Aldebaran, we should meet with intelligent beings who would express the utmost astonishment that we could ever have framed a principle leading to such a universal catastrophe.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

FEBRUARY 1891.

XI. *On the Change in the Absorption-Spectrum of Cobalt Glass produced by Heat.* *By* Sir JOHN CONROY, *Bt., M.A., Fellow and Bedford Lecturer of Balliol College, and Millard Lecturer of Trinity College, Oxford**.

SIR DAVID BREWSTER made some experiments on the influence of heat on the absorbing power of coloured media, and states in his 'Treatise on Optics' (edit. 1853, p. 174) that he "was surprised to observe that it produced opposite effects upon different glasses, diminishing the absorbing power in some, and increasing it in others." He found that the transparency of a piece of purple glass was much increased on heating, whilst that of a yellowish-green glass and of a red glass was diminished; the purple glass recovered its colour on cooling, the other two did not do so completely.

Feussner (*Fortschritte der Physik*, 1867, p. 237) made some observations on the effect of heat on the absorption-spectra of substances in solution.

No observations on the effect of heat on the transparency of solid substances for rays of different refrangibilities except those of Sir David Brewster appear to have been published, although, of course, the change of colour which borax blowpipe-beads containing certain metallic oxides undergo on cooling is well known. I therefore venture to communicate to the Physical Society some determinations which I have recently made on the changes produced by heat in the absorption-spectrum of cobalt glass.

---

* Read February 13, 1891.

The absorption-spectra of coloured glasses are not, as a rule, very characteristic, and merely show a continuous absorption extending through a considerable portion of the spectrum. Cobalt glass, however, as is well-known, has a characteristic absorption-spectrum consisting of three dark bands, in the red, yellow, and green, with a considerable amount of absorption between the first two; so that with a rather deeply coloured glass the transmitted light consists merely of the extreme red, some yellowish-green, and the blue and violet rays.

A small piece of this glass was heated by means of a medium-sized Bunsen (15 millim. tube), the glass being supported on, and nearly surrounded by, combustion-furnace tiles. It was found that in this way the glass could be heated till the edges began to soften and were visibly red, without much risk of its cracking either whilst being heated or during its subsequent cooling. An ordinary gas-jet was used as the source of light, and the light transmitted by the glass examined with a spectroscope; a small direct-vision one being used, as it was found that the changes in the spectrum were less distinctly seen with a spectroscope of greater dispersive power.

The absorption-spectrum of the cold glass is represented in fig. 1, which is drawn to an arbitrary scale, and not to one of wave-lengths. On gradually heating the glass, the absorption between the two first dark bands, those in the red and yellow, diminishes. The band in the red moves towards the least refrangible end of the spectrum, whilst those in the yellow and green retain their position, but become less distinct. Fig. 2 represents the spectrum of the hot glass.

As the glass is heated the intensity of its colour decreases; as it cools it recovers its original colour, and the absorption-spectrum changes back into that represented by fig. 1.

Fig. 1.

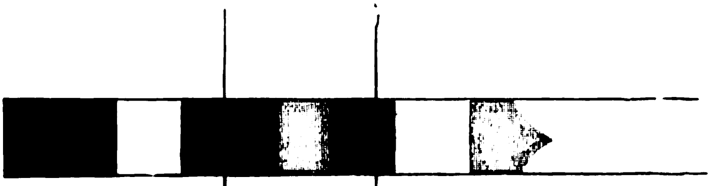
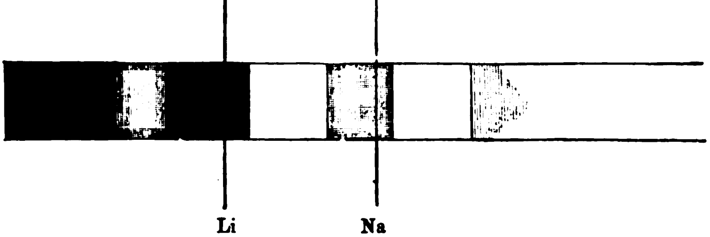


Fig. 2.



The position of the bands was measured, and the table gives the results, reduced to a scale of wave-lengths.

	Cold Glass.	Hot Glass.	Dr. Russell's Values.
Band I.	700	712	665
	636	655	632
Band II.	608	608	605
	580	583	588
Band III.	565	558	555
	indistinct	indistinct	

Dr. W. J. Russell, in a paper on the "Absorption Spectra of Cobalt Salts" (Proc. Roy. Soc. xxxii. p. 258) gives a map of the spectrum of cobalt glass drawn to a scale of wave-lengths. The position of the bands, as shown in Dr. Russell's map, is given in the third column of the above table.

The agreement between the values obtained with the cobalt glass when cold and those from Dr. Russell's map, except for the least refrangible edge of the first band, are as close as, perhaps, could well be expected, considering the small dispersion of the spectroscope used for the measurements, and the difficulty of determining the exact position of an absorption band. There is considerable difference with regard to the position of the least refrangible edge of band 1; that the position assigned to it for the particular sample of glass used in these experiments is, at least, approximately correct, is shown by the fact that the red lithium line of wave-length

670 lies within the band, both when the glass is cold and when it is hot.

These observations and those of Feussner show that the absorption-spectra of some substances vary with the temperature, as indeed might have been anticipated from the behaviour of the blowpipe-beads already referred to. In the case of solutions this may be due to the formation of different hydrates, or to the partial dissociation of the substances; but in the case of a solid substance, like cobalt glass, an actual change in the chemical constitution of the glass at a temperature considerably below its fusing-point does not seem very probable, although the well-known effects to light in causing glass which has been decolorized with manganese dioxide to become purple seems to show that such a change is not impossible.

XII. *Further Contributions to Dynamometry, or the Measurement of Power.* *By* T. H. Blakesley, *M.A., M. Inst. C.E.*[*]

Now that the advantage of using the split dynamometer to measure power, as first proposed by me more than half a decade since, seems to be at length receiving some attention through the generalization of the method when applied to transformers, proved independently by Prof. W. E. Ayrton and myself[†], it may be well to point out clearly the principles upon which such an instrument must be inserted into an electrical system to effect the measurement of a physical quantity, and the nature of the quantities which admit of such measurement.

In the first place, an exact idea must be formed of the nature of the physical quantity indicated by the reading of a dynamometer, or the angle through which the torsion-head is turned to bring the coils into a standard relative position, which is usually, but not necessarily, one in which the coils are at right angles one to the other. That position has the advantage of introducing no mutual induction in the instrument itself.

---

[*] Read February 27, 1891.

[†] To Professor Ayrton and Mr. Taylor belongs undoubtedly the credit of priority in this generalization. Professor Ayrton informed me of the *fact*, but left me to discover the *proof*.—T. H. B.

Expressed mathematically, the reading measures the quantity

$$\frac{1}{T}\int_0^T C_1C_2dt;$$

where C_1 and C_2 are the currents at a moment through the two coils, those currents being periodic or constant (one may be constant, the other periodic), and T being an interval of time at least equal to the least common measure of the periods, and so small that the index is not able to move appreciably in the interval T. The larger the moment of inertia of the moving coil, the greater the limit which may be allowed to T.

In the year 1885, when I first suggested sending different currents through the two coils of such an instrument, I called a reading taken under such circumstances the "force-reading," to distinguish it from an ordinary dynamometer-reading in the usual case of the currents being identical in the two coils. That name was suggested by the fact that $(\text{current})^2$ has for its structural formula in the electromagnetic system the same dimensions as force, omitting the dimension of permeability. This fact is shown in Sir W. Thomson's so-called current-balances, where $(\text{current})^2$ is made to produce equilibrium with a force.

But $(\text{current})^2$ has another more important meaning. When multiplied by resistance, it means power, and therefore by itself it means power per unit of resistance; and this is its true meaning independently of permeability. The dynamometer-reading is the *mean power* per unit of resistance.

If, therefore, we know the proper resistance to multiply the dynamometer-reading by, we shall be in possession of the value of the power; and it follows that appropriate dynamometer-readings must be of extreme value in measuring power.

It will thus be seen that if the physical quantity Z can be expressed for its momentary value in terms quadratic in the instantaneous currents, these terms will point out to us the appropriate places for dynamometers whose readings, being filled in in the places of those quadratic expressions, will give us the mean value of (Z). To make this perfectly clear:—

Suppose

$$Z = A\,.\,c_1^2 + Bc_1c_2 + Cc_2^2$$

at any instant; then the mean value of Z will be

$$A\,.\,{}_1D_1+B\,{}_1D_2+C\,{}_2D_2,$$

where ${}_1D_2$ is the reading of a dynamometer one of whose coils carries c_1 and the other c_2.

If Z is power, A, B, C are of the order resistance. If Z is $\overline{\text{E.M.F.}^2}$, A, B, C are of the order (resistance)^2. It is necessary that A, B, C should not be functions of the time. Hence power and $\overline{\text{E.M.F.}^2}$, the latter being merely power per unit of conductivity, are very appropriate quantities for the method.

To take the simple case of two machines working in parallel into a third inductionless circuit. The equations are

$$e_1-f=r_1c_1,$$
$$e_2-f=r_2c_2,$$
$$f=r_3c_3,$$

and $$c_1+c_2=c_3\ ;$$

where e_1 and e_2 are the total E.M.F.s of the generators in the two loops 1 and 2 (including all induction); c_1, c_2, and c_3 are the three instantaneous currents, the two former positive towards the same point of junction, the latter positive towards the other, so that $c_1+c_2=c_3$ always; and f is the potential difference at the points where the circuits join.

Then, since

$$e_1c_1=r_1c_1^2+r_3c_1c_3 \text{ or } \overline{r_1+r_3}c_1^2+r_3c_1c_2,$$
$$e_2c_2=r_2c_2^2+r_3c_2c_3 \text{ or } \overline{r_2+r_3}c_2^2+r_3c_1c_2,$$

we have power of 1st generator

$$=r_1\,{}_1D_1+r_3\,{}_1D_3, \text{ or } \overline{r_1+r_3}\,{}_1D_1+r_3\,{}_1D_2\ ;$$

power of 2nd generator

$$=r_2\,{}_2D_2+r_3\,{}_2D_3, \text{ or } \overline{r_2+r_3}\,{}_2D_2+r_3\,{}_1D_2,$$

where D refers to a dynamometer-reading.

Here we appear to require four dynamometers; but the expression for the instantaneous power may be written in the 2nd form given, which necessitates only three dynamometers. Either generator here becomes a motor if the second term as given above has changed sign, and is of greater numerical value than the first term, which is necessarily positive.

The expressions for determining the mean $\overline{\text{E.M.F.}}^2$ of the machines are:—

$$\left.\begin{aligned}(\text{mean } e_1^2) &= r_1^2\,{}_1\mathrm{D}_1 + r_3^2\,{}_3\mathrm{D}_3 + 2r_1r_3\,{}_1\mathrm{D}_3,\\ (\text{mean } e_2^2) &= r_2^2\,{}_2\mathrm{D}_2 + r_3^2\,{}_3\mathrm{D}_3 + 2r_2r_3\,{}_2\mathrm{D}_3,\end{aligned}\right\}\text{ taking five dynamometer-readings.}$$

But this can be simplified, as in the formulæ for the powers, thus:—

$$\begin{aligned}e_1 = r_1c_1 + r_3c_3 &= \overline{r_1 + r_3}c_1 + r_3c_2,\\ e_2 = r_2c_2 + r_3c_3 &= \overline{r_2 + r_3}c_2 + r_3c_2;\\ \therefore\ e_1^2 &= \overline{r_1 + r_3}^2\,{}_1\mathrm{D}_1 + r_3^2\,{}_2\mathrm{D}_2 + 2\cdot\overline{r_1 + r_3}\cdot r_3\,{}_1\mathrm{D}_2,\\ e_2^2 &= \overline{r_2 + r_3}^2\,{}_2\mathrm{D}_2 + r_3^2\,{}_1\mathrm{D}_1 + 2\,.\,\overline{r_2 + r_3}\,.\,r_3\,{}_1\mathrm{D}_2;\end{aligned}$$

in which expressions there are only three dynamometer-readings, and these the same three as for giving the two powers.

It is clear that $-r_3c_3c_2$ is the power doing work upon the second circuit; for it is equal to $-fc_2$ at any moment; $\therefore$ $-r_3\,{}_2\mathrm{D}_3$ is the mean power expended in the second circuit.

This is quite independent of the nature of the apparatus in the second circuit, which may contain any or all of the following:—

A perfect or absorbent condenser,
An electromagnet,
A decomposing-cell,
A vacuum-tube,
A motor-circuit,
A transformer-circuit,
A generating-circuit,
A welding-machine,
A tuning-fork, or other make and break.

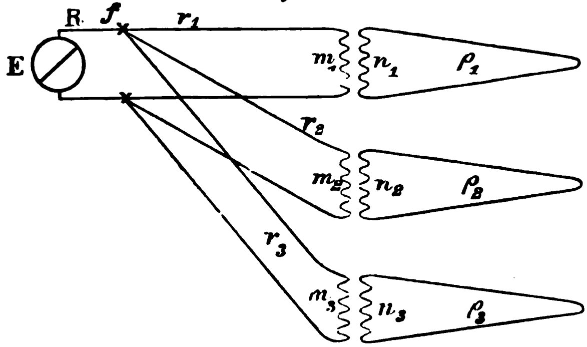
Should the apparatus render it undesirable to have the current c_2 passed through the dynamometer, we may write

$$-r_3c_3c_2 = -r_3c_3(c_3 - c_1) = r_3c_1c_3 - r_3c_3^2,$$

or $$\text{Mean Power} = r_3\{{}_1\mathrm{D}_3 - {}_3\mathrm{D}_3\}.$$

It was by this means that I suggested to Mr. Swinburne he might measure the dielectric hysteresis of his condensers. It would only take two dynamometers, as is seen.

The Case of a Machine playing into a Number of Parallel Transformers.



Let E be the E.M.F. of the machine (including all induction);
R its resistance ;
f the P.D. where the circuits become parallel ;
r the resistance of the primary from this point ;
m the number of turns in the primary coil ;
n the number of turns in the secondary coil ;
ρ the resistance of the secondary coil.
C is the current through the generator ;
c is the current in the primary;
γ is the current in the secondary;
N the number of magnetic lines in the core.

Numerical subscripts must be applied where required for the various circuits.

Then

$$\left.\begin{array}{l} E-f=CR, \\ \left.\begin{array}{l} f-m\dfrac{dN}{dt}=cr, \\ n\dfrac{dN}{dt}=\gamma\rho, \end{array}\right\} \therefore\ f=cr+\dfrac{m}{n}\gamma\rho\ ; \end{array}\right\} \therefore\quad E=CR+cr+\frac{m}{n}\gamma\rho;$$

multiplying f by c, $fc=c^2r+\rho\dfrac{m}{n}\gamma c$,

which gives the power in each transformer circuit, and indicates the appropriate positions of the two instruments, viz. one with both coils in the primary, and one split between the primary and the secondary.

$\rho\dfrac{m}{n}\gamma c$ is the power heating transformer and secondary,

$\rho\gamma^2$ is the power heating the secondary ;

$\therefore\ \rho\frac{m}{n}\gamma c - \rho\gamma^2$ is the power heating the core, being the result of hysteresis and Foucault currents.

Multiply E by C, and we have the total power, which is

$$\mathrm{R}\,.\,\mathrm{C}^2 + r\mathrm{C}c + \frac{m}{n}\rho\,.\,\mathrm{C}\gamma.$$

Thus the whole power consumed in such a system may be measured by means of three dynamometers judiciously applied to the main current and those in *a single* branch.

This, of course, can be applied to measure the power at work in a system of parallel transformers by means of a home transformer, whose resistances may be kept fixed or at least accurately determined.

It may also be used to measure the electric power of a *welding* transformer, into whose secondary it is inexpedient, if not impossible, to introduce any extra resistance.

In this case a very high power would be rendered measurable by increasing the resistances in the circuits of the parallel system. The power consumed in the parallel system being, as indicated, known, can be deducted from the total.

Transformers in Series.

Let E be the E.M.F. of the machine;
r_1 be the resistance of the 1st primary;
r_2 „ „ 1st secondary and 2nd primary;
r_3 „ „ 2nd „ 3rd „
N_1 be the magnetic lines in 1st core;
N_2 „ „ 2nd „
$\left.\begin{matrix} m_1 \\ n_1 \end{matrix}\right\} =$ turns in $\left\{\begin{matrix} \text{primary} \\ \text{secondary} \end{matrix}\right\}$ of 1st transformer;
$\left.\begin{matrix} m_2 \\ n_2 \end{matrix}\right\} =$ „ „ 2nd „
c_1 = current in 1st section;
c_2 = „ 2nd „

Then

$$\mathrm{E} - m_1\frac{d\mathrm{N}_1}{dt} = r_1c_1;$$

$$n_1\frac{d\mathrm{N}_1}{dt} - m_2\frac{d\mathrm{N}_2}{dt} = r_2c_2,$$

$$n\,\frac{d\mathrm{N}_2}{dt} - m_3\,\frac{d\mathrm{N}_3}{dt} = r_3c_3,$$

&c., &c.

$$\therefore\ \mathrm{E} = r_1c_1 + \frac{m_1}{n_1}\,r_2c_2 + \frac{m_1m_2}{n_1n_2}\,r_3c_3 + \frac{m_1m_2m_3}{n_1n_2n_3}\,r_4c_4 + \&c.$$

The series has $\overline{q+1}$ terms when there are q transformers. The square of this expression will have every term quadratic in current, and be thus amenable to dynamometer treatment.

The value for one transformer has already been given by me.

If we multiply through by c_1, we have the total power c_1E given in suitable terms.

For one transformer,

$$c_1E = r_1c_1^2 \quad + \frac{m}{n} r_2c_2c_1;$$

or power

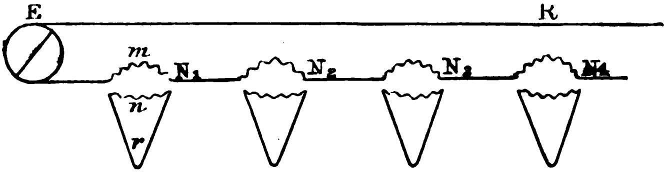
$$= r_1\,{}_1D_1 + \frac{m}{n} r_2\,{}_1D_2,$$

the first term heating the primary; $r_2\,{}_2D_2$ is the power heating the secondary.

Therefore the power involved in warming the core by the magnetic changes is

$$\frac{m}{n} r_2\,{}_1D_2 - r_2\,{}_2D_2 \quad = r_2\left\{\frac{m}{n}\,{}_1D_2 - {}_2D_2\right\}.$$

The Case of Transformers with Primaries in Series.



E is the E.M.F. of the generator;
N the number of magnetic lines in the core;
m the number of turns in the primary of a transformer;
n the number of turns in the secondary of a transformer;
C the current in the primary;
R its resistance;
c the current in the secondary;
r its resistance.

Numerical subscripts being added where required.

Then

$$E - m_1\frac{dN_1}{dt} - m_2\frac{dN_2}{dt} - m_3\frac{dN_3}{dt} - \&c. = CR.$$

$$n_1\frac{dN_1}{dt} = c_1r_1\,;$$

$$n_2\frac{dN_2}{dt} = c_2r_2\,;$$

$$n_3\frac{dN_3}{dt} = c_3r_3\,;$$

$$\&c. \qquad \&c.$$

$$\therefore\ E = CR + \frac{m_1}{n_1}c_1r_1 + \frac{m_2}{n_2}c_2r_2 + \frac{m_3}{n_3}c_3r_3 + \&c.$$

The square of this will be quadratic in c, and the terms will indicate the proper places for dynamometers. The total power is

$$EC = C^2R + r_1\frac{m_1}{n_1}Cc_1 + r_2\frac{m_2}{n_2}Cc_2 + \&c.$$

The first term heats the primary, and each succeeding term indicates the power employed in heating a secondary and a core corresponding with it. Both in this case and in the case of parallel transformers it appears that the power heating the core and its secondary is indicated by one dynamometer-reading alone, one coil being in the primary and the other in the secondary; the reading requiring multiplication by the ratio of the coil-turns in the primary to those in the secondary, and by the secondary resistance; *i. e.* this power

$$= r_2\,.\frac{m}{n}\,._1D_2.$$

Does not this indicate the direction which efforts should take to effect a really fair mode of measuring Electrical Energy supplied?

In some cases a single instrument might be used, even if the formula indicated two terms, to obtain the required measurement. Suppose, for instance, that the formula had two terms $(c_1^2 - c_1c_2)$. This may be written $c_1(c_1 - c_2)$; and it is clear that if we had two fixed similar coils both in one plane, and made to carry c_1 and c_2 *reversed*, respectively, and if the movable coil were made to carry c_1, then the indications of the instrument would give the required measurement. It might be possible to multiply such coils, and vary their turns and position so as to meet any case, if desirable. The method is merely indicated here.

From what I have said, it will not surprise those who have followed me that in questions of power I recommend that quotations should be made of mean $\overline{\text{current}^2}$, as indicated by dynamometers, and that attempts at giving mean current in amperes, with reversing currents, should be given up. What a quantity of pains has been taken to make voltmeters give indications proportional to volts! The merit of a difference of potential is as its square, and that of cells as the square of their E.M.F. Give me twice the E.M.F., you quadruple my power of doing work by its means. To reduce readings to give the square root of mean square is doubly wrong. It is a ridiculous attempt to reach a useless quantity, and, further, gives one the trouble of squaring back again.

XIII. *Proof of the Generality of certain Formulæ published for a Special Case by* Mr. Blakesley. *By* Prof. W. E. AYRTON, *F.R.S., and* J. F. TAYLOR[*].

I.

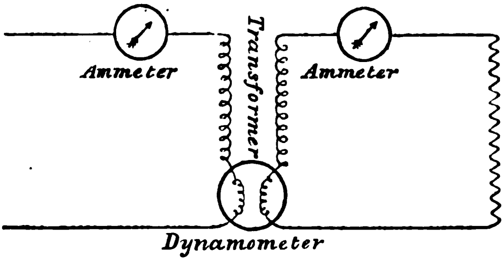
IN May 1888 Mr. Blakesley described before this Society[†] a very interesting method of testing the power given to the primary coil of a transformer based on the employment of three dynamometers. The method, however, really only requires two alternate current ammeters and one dynamometer with two coils. The ammeters are placed respectively in the primary and secondary circuits of the transformer, while the two coils of the dynamometer are electrically separated from one another, one being placed in the primary and the other in the secondary circuit of the transformer, as shown in the figure.

Mr. Blakesley in his paper assumed that the three instruments were not graduated directly; that is to say, that constants had to be employed in each case to reduce the arbitrary scale-readings to absolute measure. We will, on the contrary, suppose the instruments to be graduated as they ought to be,

---

[*] Read February 27, 1891.

[†] Proc. Phys. Soc. ix. p. 286; Phil. Mag. [5] xxvi. p. 34.



so that if D_p, D_s, and D_{ps} be the readings of the three instruments, they are equal respectively to the square root of the mean square of the primary current, the square root of the mean square of the secondary current, and the mean product of the two currents.

With this definition Mr. Blakesley proved geometrically that the watts given to the primary coil of the transformer were equal to

$$pD_p^2 + \frac{P}{S} sD_{ps};$$

where p and s are the resistances, in ohms, of the primary coil and the whole secondary circuit respectively, and P and S are the numbers of windings in the primary and secondary coils of the transformer.

The formula is a simple one, and the values of the expressions in it are fairly easy to obtain experimentally. The proof of the formula, however, as given by Mr. Blakesley, was based on the following assumptions:—

1. The variations of the primary and secondary currents are harmonic.

2. The variation of the magnetism of the core is harmonic.

3. The magnetic stresses produced in the iron core by the currents in the primary and secondary coils are directly proportional to the ampere-turns in these two coils.

4. Each turn in each coil embraces at any moment the same number of lines of force.

5. The secondary circuit outside the transformer is non-inductive.

II.

In the spring of last year, 1890, Mr. Wightman, one of the third-year students of the Central Institution, showed that an analytical method for measuring the efficiency of a transformer, which had been described in one of the lectures at the college, could by a slight transformation be employed to prove the generality of Mr. Blakesley's formula given above.

The proof is quite simple, and shows that the formula in question is true whatever function the currents or the magnetism be of the time, and whatever amount of hysteresis or magnetic lag may exist. In fact the proof is independent of Mr. Blakesley's assumptions Nos. 1, 2, and 3, mentioned above.

We have delayed the publication of this proof until we had used the method for a lengthy series of experiments on a transformer kindly lent us by Mr. Mordey, and which has occupied the students for many months. But, in order that others like ourselves should be able to use Mr. Blakesley's formula with confidence, and without having any longer the fear that if the sine law were not true, or if much hysteresis existed, their calculations made by means of this formula from experimental results might be very wrong, we communicated nearly a year ago to Mr. Blakesley and to others interested in the matter the fact that we had proved mathematically that the two ammeters and one dynamometer method of measuring power was generally true.

Let V_p be the P.D. in volts at the terminals of the primary-coil at any moment *t*.

Let A_p be the primary current, in amperes, at the same moment.

Let A_s be the secondary current, in amperes, at the same moment.

Let *n* be the number of lines of force passing through one convolution at that moment.

Then

$$V_p = pA_p + \frac{1}{10^8} P \frac{dn}{dt},$$

$$sA_s = \frac{1}{10^8} S \frac{dn}{dt};$$

$$\therefore \quad V_p = pA_p + \frac{P}{S} sA_s.$$

$$\therefore A_pV_p = pA_p^2 + \frac{P}{S} sA_pA_s.$$

If T be the time of one complete cycle,

$$\frac{1}{T}\int_0^T A_pV_p dt = \frac{p}{T}\int_0^T A_p^2 dt + \frac{P}{S}\frac{s}{T}\int_0^T A_pA_s dt.$$

But the expression on the left-hand side is the mean watts given to the primary coil, and the expression on the right-hand side is simply

$$pD_p^2 + \frac{P}{S} sD_{ps};$$

D_p, as already explained, being the reading of the alternate current ammeter in the primary circuit, and D_{ps} the reading of the dynamometer having one coil in the primary and the other coil in the secondary circuit.

The following is another general proof of the same formula :—

The watts given to the primary coil of a transformer are spent partly in heating the primary coil, and partly in doing work against the back electromotive force set up by the varying magnetism of the core.

The watts spent in heating the primary coil are of course pD_p^2; while the watts spent in doing work against the back electromotive force are at any moment $A_p \times$ the back E.M.F.

The E.M.F. generated in the secondary circuit by the same variation of magnetism of iron is at any moment sA_s; therefore the back E.M.F. in the primary coil at the same moment must be $\frac{P}{S} sA_s$.

Consequently the total watts given to the primary coil are

$$pD_p^2 + \frac{P}{S}\frac{s}{T}\int_0^T A_pA_s dt;$$

$$pD_p^2 + \frac{P}{S} s\, D_{ps}, \text{ as before.}$$

III.

This expression, as Mr. Blakesley points out, may be written as follows :—

$$pD_p^2 + sD_s^2 + s\left\{\frac{P}{S}D_{ps} - D_s^2\right\};$$

and since $pD_p^2 + sD_s^2$ are the watts employed in heating the primary coil and the whole of the secondary circuit, it follows that

$$s\left\{\frac{P}{S}D_{ps} - D_s^2\right\}$$

are the watts employed in heating the core on account of hysteresis or magnetic lag ; a result now proved true independently of all assumptions as regards the sine law, or the magnetic stress being directly proportional to the ampere-turns, &c.

IV.

Since the portion of the secondary circuit outside the transformer is non-inductive and has a resistance s' say, the watts developed in it are of course $s'D_s^2$. Consequently the efficiency of the transformer is

$$\frac{s'D_s^2}{pD_p^2 + \frac{P}{S}sD_{ps}}.$$

V.

Using the various assumptions already referred to, Mr. Blakesley arrived, by means of a geometrical proof, at a formula for measuring the mean square of the P.D. at the terminals of the primary coil by means of the two ammeters and the dynamometer.

The following *general* proof of this formula is very much simpler than the proof for only a special case which Mr. Blakesley gives, and furnishes a very good example of the fact that sometimes it is more easy to give an analytical proof which is true independently of any assumptions about the harmonic law, &c. than to give a geometrical proof which is only true when these suppositions hold.

$$V_p = pA_p + \text{the back E.M.F. at the moment}$$

$$= pA_p + \frac{P}{S} sA_s\,;$$

$$\therefore\ \frac{1}{T}\int_0^T V_p^2 dt = \frac{p^2}{T}\int_0^T A_p^2 dt + \frac{P^2 s^2}{S^2\,T}\int_0^T A_s^2 dt + 2\frac{P}{S}\frac{ps}{T}\int_0^T A_p A_s dt\,;$$

that is, the mean square of the P.D. at the terminals of the primary coil equals

$$pD_p^2 + \frac{P^2}{S^2} s^2 D_s^2 + 2\frac{P}{S} psD_{ds},$$

which is the formula given by Mr. Blakesley.

From the preceding it follows that Mr. Blakesley's expressions for the watts given to the primary coil of a transformer, for the efficiency of the transformer, and for the mean square of the P.D. at the terminals of the primary coil, are true irrespectively of any assumptions as to the functions the E.M.F.s, the currents, or the magnetic flux are of the time as well as of any assumptions as to the presence or absence of hysteresis or magnetic lag.

This being the case, the application of this two ammeters and dynamometer method for measuring power in other cases than those already treated of is worthy of careful consideration.

XIV. *On the Variation of Surface-Tension with Temperature.*

By Prof. A. L. Selby, *M.A., University College, Cardiff* *.

Mendelejeff speaks of an ideal liquid as characterized by two conditions :—

(1) $V_t = V_0/(1-kt)$, V_t being the specific volume at $t°$ C.

(2) $T_t = T_0(1-at)$, T_t being the surface-tension at $t°$ C.

I believe that the following proof shows that all liquids satisfy Mendelejeff's second condition.

Let unit mass of liquid have a constant volume, but variable surface S, and temperature t.

---

* Read March 20, 1891.

In a small change of the variables, the heat absorbed is

$$dH = k\,dt + l\,dS,$$

k being the specific heat at constant volume, l the latent heat of extension.

The external work done on the film is

$$dW = T\,dS.$$

Therefore the gain of intrinsic energy is

$$dH + dW = k\,dt + (l+T)\,dS.$$

This is a perfect differential.

Therefore

$$\frac{dk}{dS} = \frac{d(l+T)}{dt}.$$

Also $\frac{dH}{t}$ is a perfect differential.

Therefore

$$\frac{d}{dS}\frac{k}{t} = \frac{d}{dt}\frac{l}{t}.$$

Therefore

$$\frac{1}{t}\frac{dk}{dS} = \frac{1}{t}\frac{dl}{dt} - \frac{l}{t^2}.$$

Therefore

$$\frac{dT}{dt} = -\frac{l}{t}.$$

And

$$\frac{d^2T}{dt^2} = -\frac{1}{t}\frac{dl}{dt} + \frac{l}{t^2} = \frac{1}{t}\frac{dk}{dS}.$$

Now k does not depend on the surface unless the film is very thin.

Therefore

$$\frac{d^2T}{dt^2} = 0.$$

And

$$T = c - bt,$$

where c and b may be functions of the specific volume.

That b does not depend on the specific volume may be shown as follows.

Let the liquid be maintained at constant temperature, and have a volume v and surface S; and let L be the latent heat of dilatation at constant surface.

Put the liquid through a cycle consisting of two isometrics v, $v+dv$, and two lines of constant surface S, S$+d$S.

Since the cycle is reversible and the temperature is constant, the heat absorbed is zero.

Therefore

$$\frac{dl}{dv}=\frac{d\mathrm{L}}{d\mathrm{S}}=0.$$

Therefore l is independent of v, and so is b since $l=bt$.

Thus the latent heat of extension is proportional to the absolute temperature. This agrees with a hypothesis of Clausius (Phil. Mag. 1862, vol. xxiv.).

It has been shown that T can be expressed in the form

$$f(v)-bt.$$

We shall show that it can also be written $\phi(p)-bt$.

For let the pressure of the liquid remain constant while the surface, volume, and temperature vary.

Then

$$d\mathrm{H}=\mathrm{K}dt+ld\mathrm{S},$$

K being the specific heat at constant pressure and l having the same meaning as before, for the latent heat of extension at constant temperature and volume is also the latent heat of extension at constant pressure (and temperature).

The external work done on the liquid is

$$d\mathrm{W}=\mathrm{T}d\mathrm{S}-p\frac{dv}{dt}dt,$$

p being regarded as constant in forming $\frac{dv}{dt}$.

Therefore,

$$d\mathrm{H}+d\mathrm{W}=\left(\mathrm{K}-p\frac{dv}{dt}\right)dt+(l-\mathrm{T})d\mathrm{S}.$$

Since this is a perfect differential,

$$\frac{d}{dt}(l+\mathrm{T})=\frac{d}{d\mathrm{S}}\left(\mathrm{K}-p\frac{dv}{dt}\right)=0,\ \text{except for a very thin film.}$$

Now $\qquad l=bt.$

Therefore $\qquad \mathrm{T}=\phi(p)-bt.$

But the two expressions $\phi(p)-bt$ and $f(v)-bt$ can only be identical at all temperatures if $f(v)$ and $\phi(p)$ are both equal to a constant c.

Therefore

$$T=c-bt,$$

where c and b are constant.

It appears then that the surface-tension of a liquid is independent of the pressure and depends only on the temperature, unless the film is very thin.

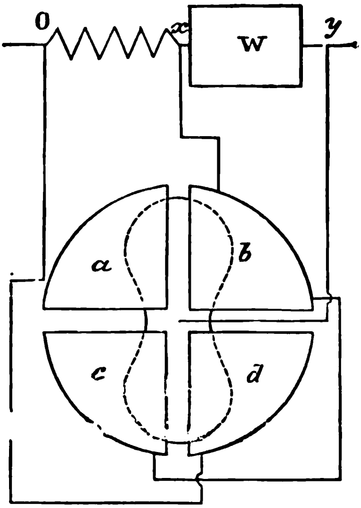
The critical temperature is c/b, and can be found by determining the surface-tension at two very different temperatures.

XV. *The Electrometer as a Wattmeter.*

By J. SWINBURNE[*].

IN 1881, when M. Joubert published his experiments on a Siemens machine, in the course of which he had used a Thomson or Mascart electrometer as a voltmeter, Professors Ayrton and Fitzgerald simultaneously proposed to use the quadrant electrometer as a wattmeter.

Fig. 1.



The ordinary method of arranging the instrument is shown in fig. 1. The resistance is wound so as to be non-inductive;

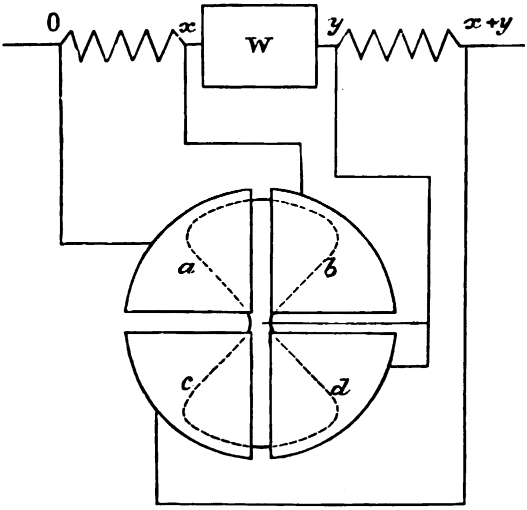
---

[*] Read March 6, 1891.

and the power to be measured is spent in the apparatus marked W. If the fall of potential between x and y is very great in comparison with that over the resistance, the instrument reads like a charged electrometer, and it may be taken to read in watts. In practice such conditions do not occur; for if the resistance is made low the instrument is not sensitive enough, and if it is made high the electrometer no longer reads in watts. If the instantaneous pressures are 0, x, and y, the force exercised by the first quadrant a in the positive direction is $k\ y^2$, where k is the constant of the instrument. This constant may be omitted, and the force denoted by y^2. The force exercised by b is $-(y-x)^2$; by $c, -(y-x)^2$; and by d, y^2. The total force is thus $2\ (2y-x)\ x$. The needle contact may now be moved from y to x and another reading taken. This is of course $2\ x^2$. Subtracting this from the first reading, we get $4\ (y-x)\ x$, which is the power taken by W. This is the arrangement that was adopted by Dr. Hopkinson in his measurements of the Gaulard and Gibbs transformer in 1884. During the first reading the instrument is really two idiostatic voltmeters, one being in shunt to both the resistance and W, and the other in shunt to W alone. During the second reading it is a voltmeter in shunt to the resistance alone.

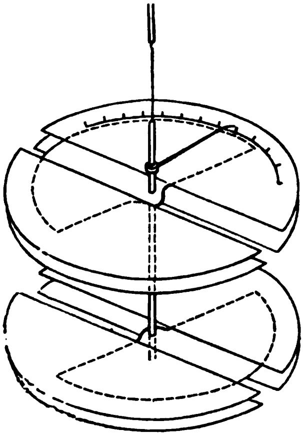
Professor Ayrton arranges the instrument so that the quadrants are in shunt to W, and the needle is first connected to x and a reading taken, and the needle is then connected to 0 and a second reading taken. During the first reading, the instrument is a voltmeter in shunt to W, and during the second it is two voltmeters, one in shunt to the resistance and one in shunt to the whole circuit. The difference again gives the power spent in W. Mr. Smith uses a discharge-key for making a change quickly. The discharge-key can be used in either arrangement, but the first is more accurate. For instance, suppose $y-x$, the pressure on W, is 2000 volts, and x, the pressure on the resistance, 20, and suppose the instrument one per cent. low at one reading. The first reading by the second method is 8,000,000, the second 8,160,000. Suppose the latter is read one per cent. low, viz. 8,078,400, the resulting determination is 78,400 instead of 160,000; that is to say, the power is more than 100 per cent. greater than that given by the instrument. In the arrangement used by Dr. Hopkinson, however, the readings are 160,800 and 800, so a misreading of one per cent. in the first makes one per cent. error only, and in the second makes no sensible difference.

Fig. 2.



The quadrant electrometer may, however, be arranged so as to read power directly without any change of connexions. This is shown in fig. 2. A second resistance equal to the first is put on the other side of W, and the quadrants are connected up as shown. The first quadrant *a* then has a force

Fig. 3.



y^2, the second b, $-(y-x)^2$. So far the instrument is like that shown in fig. 1, but with only half its quadrant utilized. Quadrant c has a force $-x^2$, and d is inactive; so that quadrant c makes the correction for which the second reading was necessary in fig. 1, so the instrument can be graduated in watts.

There may be some slight error due to the whole needle being drawn into the quadrants a and b, and out of c and d. This is obviated by making the needle of such a shape that there is no appreciable end pull, as shown in fig. 2.

Fig. 3 shows a form for a direct-reading instrument with a pointer or index. Instead of quadrants it has half-disks, like the Blondlot and Curie electrometer, but, unlike it, this form has four pairs of half-disks. The Blondlot and Curie form also reads power directly, but the needle is made in two insulated portions, and needs two metallic connexions, and this gives rise to mechanical troubles and loss of sensitiveness. The instrument shown in fig. 3 has the needle all in one piece, and the disks can be so far apart that errors from variations of height of the needle due to variations of the length of the fibre do not become serious. A long suspension of phosphor-bronze wire is, however, preferable for most purposes.

XVI. *Alternating and Experimental Influence-Machine.*

By Mr. JAMES WIMSHURST, *Member of Council*[*].

I HAVE pleasure in bringing to your notice a new form of influence-machine which is self-exciting, notwithstanding that when at work its electrical charges alternate during each revolution.

In order that you may readily follow the action of the machine when at work I will first describe its construction in all its details.

It consists of a base or frame, from the sides of which rise the standards to carry the spindle and boss for the rotating disk, suitable driving gear being fixed thereto.

In the same plane as the rotating disk is fixed a square wooden frame having the necessary holes, plugs, and clamps, by means of which the inductor-plates are held in position.

[*] Read April 17, 1891.

The rotating disks are cut from ordinary window glass, and are coated with shellac; they are 16 inches in diameter; one of them has no metal upon it, the next has four medium-sized tin-foil sectors upon each of its sides, the last of this series has 16 sectors upon each side; other disks have from 2 to 4 sectors of large size upon them; another disk has four large sectors upon each side, so placed that the sectors upon one side cover those upon the other side; another has four very narrow sectors upon each side, and another has 16 sectors all upon the one side.

The inductor-plates are squares of glass measuring 9¾ inches; one corner of the plate is cut away to admit the spindle and the boss. They are coated with shellac, and upon one side of each of them is a tin-foil patch, and a suitable device for holding the rod and the brush.

Two of these inductor-plates are mounted at the diagonal corners, upon one side of the wooden frame, and two upon the other side of the frame; those at the front of the machine are at the lower right-hand corner and the upper left-hand corner; those at the back of the machine are at the upper right-hand corner and the lower left-hand corner: the rotating disk is therefore covered for one half of its surface upon both of its sides.

The brushes are made of fine brass wire, and the brush-holders are brass rods, bent to a form to admit of the brushes touching the rotating disk at a point opposite to the middle of the next following induction-plate; this arrangement supplies two brushes to each side of the disk, and the brushes when in place are situated 180° asunder.

The several parts of the machine are interchangeable, and by means of the varied combinations many experiments can be made; in fact its combinations include nearly every type of electrical influence-machine.

The prominent results obtained from it are:—(1) That glass disks which have no metal upon them are freely self-exciting. (2) That the freedom to self-excitement increases about proportionally to the number of sectors. (3) That the quantity of electricity decreases with the amount of metal upon the disk—whether the amount be in the greater number of sectors, or the increased size of sector (chiefly the latter).

The only tests as to when the alternations occur which I have been able to make were made by means of a sensitive arrangement of light paper disks, suspended by fine wire. When this apparatus is connected to one of the inductors, and the glass disk turned very slowly, the alternations are seen to occur with each $\frac{3}{4}$ revolution of the disk; when the disk is turned much faster, then the alternations occur too rapidly for the paper disks to respond, and they hang motionless and nearly together.

It is not possible to obtain any sensible charge in a Leyden jar, although the electricity may be clearly seen as a stream between the jar and the inductor.

Another series of combinations may be made by removing the two inductor-plates from the back of the machine and substituting an insulating arm extending across the disk, it having wire brushes at its ends, the brushes being connected metallically with terminal balls. When this combination is in use the charges no longer alternate unless the terminal balls are separated beyond the sparking distance. The plain glass disk without metal sectors, and also the disks having large sectors upon them, are no longer self-exciting. The glass disk having 16 sectors upon one side is not self-exciting when placed so that the sectors touch the brushes of the inductors, but when placed so that the sectors touch the brushes of the insulating arm it then becomes freely self-exciting. All disks having medium-sized sectors upon each side are freely self-exciting.

I have noted many of these results in tabular form.

I now feel uncertain as to whether I should end my paper at this stage, or whether I should extend it into the region of opinion. If I remain silent I am sure the cause of the electrical action will be dealt with by abler minds than my own. On the other hand, I feel equally sure that many will wish me to indicate a working hypothesis; therefore, and by way of suggestion, I will add what seems to me to be a reasonable explanation.

As to the initial charge, I think it may be accepted that all bodies behave as though they possessed a film of electricity over their surfaces, and that when two or more of these bodies are brought together this normal electrical condition is upset,

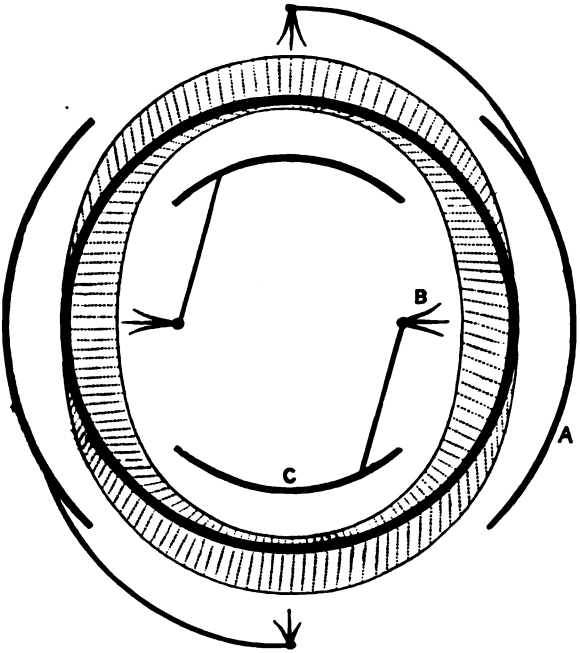
Table showing Results of Combinations.

Disks.	Four Large Inductors.	Four Small Inductors.	Two Inductors and Rod.
Plain glass and no metal.	Is self-exciting with about 4 revolutions. Alternations occur each ¾ revolution.	Is self-exciting but requires care. Alternations occur each ¾ revolution.	Is not self-exciting but requires little assistance. Constant current while not overtaxed; gives large current.
Two metal sectors each side.	Is self-exciting. Alternations occur after each two sectors pass brush.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Constant current.
Four metal sectors each side.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Constant, but gives less current.
Eight metal sectors each side.	Is more freely self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Constant current.
Sixteen metal sectors each side.	Is freely self-exciting with 3 revolutions. Alternations occur each ¾ revolution.	Is more freely self-exciting. Alternations occur each ¾ revolution.	Is freely self-exciting. Constant, but gives less current.
Sixteen sectors on one side.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting. Alternations occur each ¾ revolution.	Is self-exciting when sectors are next the rod. Is not self-exciting when sectors are next inductors. Constant in either case.

Note.—The principal features are:—1. That glass disks with no metal upon them are capable of self-excitement. 2. That the self-excitement increases about proportionally to the increase in the number of sectors. 3. That the quantity of electricity decreases about proportionally to size of sectors and also their number.

for the electricity which was upon their near surfaces is repelled to the outer or bounding surface—it is this excess charge upon the bounding surface, minute it may be, which constitutes the first charge.

In respect to the electrical action which takes place in this particular machine, I will endeavour by the help of the small diagram to explain.



Assuming that we have obtained an initial charge and that it is brought to the inductor marked A, then, neglecting the changes which take place between the two bounding surfaces, we obtain by induction an excess of electricity upon the far surface B of the rotating disk: this excess in its turn is conveyed by the brush and its holder to the next inductor C, which in its turn repeats the operation, and so on with each inductor, for the inductors are, as you see, situated alternately, the first upon one side, and the next upon the other side of the rotating disk. These excesses of electricity seated upon the rotating disk, opposite to the inductors, may be viewed as wave-crests, while the corresponding depressions are under the surface of the inductor. All that is done by the machine is to produce this wave-action in the electrical coating or film; for there is no metallic connexion between inductor and inductor, nor between the machine and the earth: moreover, all the inductors are charged with electricity of one sign, although, probably, the potential in one inductor may be slightly different to that in the other.

The alternations are possibly caused by the repulsion between the electrical charges upon the disk and the inductor, and the consequent slipping of the electrical film upon the inductor in such manner as to produce a break in the phase of the wave in relation to the brushes.

XVII. *Some Points in Electrolysis.* *By* J. Swinburne*.

Let a cell be considered, with its external circuit closed through a resistance so high that in comparison the internal resistance is negligible. The cell discharges, and in passing from one pole to the other each coulomb does a certain quantity of work proportional to the difference of potential of the poles, E. This is really a mere definition: if difference of electric potential between two points is defined as what is measured by the work done on unit quantity of electricity passing from one to the other, the work done is numerically equal to E. The work done on the coulomb passing onward through the cell to regain its original position is equal to the work done when it passes through the external resistance. This work done on the coulomb may be supplied at the expense of chemical energy, or there may be local cooling somewhere. There may be chemical energy supplied at some points, and absorbed at others, and heat may disappear at some points, and may be evolved at others, but the algebraical sum comes out equal to E. Call the poles of the cell p and n, the positive pole being, for example, platinum and the negative such a metal as zinc, and suppose the electrolyte to be one homogeneous fluid. Part of the work done on the coulomb in passing from n to the electrolyte may be supplied chemically and may be called E_{nc}, another part may be supplied by local cooling E_{nh}, c standing for chemical and h for heat. If there is really local heating this will be negative. There is no work done by the homogeneous electrolyte, and the work done on it, in overcoming its resistance, is assumed to be inappreciable. The work done on the coulomb passing from the electrolyte to the plate forming the pole p is similarly $E_{pc}+E_{ph}$. We have thus

---

* Read March 20, 1891.

$$E=E_{pc}+E_{nc}+E_{ph}+E_{nh}.$$

First assume that the chemical work done is not dependent on the temperature of the cell. Let it discharge one coulomb at temperature θ_1 and do work equal to E_{θ_1} joules ; E_{θ_1} being the electromotive force of the cell at that temperature. Of this work $E_{nc\theta_1}+E_{pc\theta_1}$ is supplied by chemical changes, and $E_{nh\theta_1}+E_{ph\theta_1}$ by cooling of the cell. Let the cell now be heated to the temperature θ, and let it be treated as a secondary element, and charged with one coulomb. The work done upon the cell is then E_θ ; so

$$E_\theta=E_{nc\theta}+E_{pc\theta}+E_{nh\theta}+E_{ph\theta}.$$

The chemical work $E_{nc\theta}$ and $E_{pc\theta}$ is common to both processes. If E_θ were equal to E_{θ_1}, the same work would be done at the two temperatures, or $E_{nh\theta_1}+E_{ph\theta_1}$ would be equal to $E_{nh\theta}+E_{ph\theta}$.

But on letting the temperature of the cell fall to θ_1 again $\frac{\theta-\theta_1}{\theta}(E_{nh\theta}+E_{ph\theta})$ is available for external work ; so that we should have perpetual motion. E_θ must therefore be greater than E_{θ_1}, so that a margin is allowed for the available work. We thus, by simple reasoning, arrive at Helmholtz's equation,

$$E=E_{nc}+E_{pc}+\theta\frac{dE}{d\theta}.$$

The electromotive force needed to do the chemical work may also vary with the temperature. For instance, if the chemical changes involved in discharging are brought about in a calorimeter at different temperatures, different heats may be evolved. Suppose E_{nc} and E_{pc} are less at a high temperature, and suppose at the lower temperature θ_1 there are no Peltier effects at the plates, so that, for that temperature at least, the cell obeys Sir William Thomson's law. If the cell could be charged at θ with a low electromotive force of $E_{c\theta}$, needed to bring the chemical changes about, and then discharged at θ, with a higher electromotive force, perpetual motion would be obtained. There must therefore be an absorption of electric energy at θ in addition to that needed to produce the chemical work. This must be liberated as heat, and some of this must be available; the cell must therefore not only have a Peltier effect at the higher temperature, but though the chemical work done at that temperature is less, the electromotive force must actually be greater. By taking a cell whose E_c varies with the temperature round the usual Carnot cycle, we get

$$\frac{dE}{d\theta} = \frac{E_h}{\theta} - \frac{dE_c}{\theta},$$

which shows that the temperature-coefficient of the cell is not affected by the variation of chemical work with the temperature. The term $-dE_c/\theta$ is the remains of the extra work that had to be allowed in compensating for the reduction of the chemical work in the example just taken. Cutting it out we get

$$E_h = \theta \frac{dE}{d\theta},$$

or

$$E = E_c + \theta \frac{dE}{d\theta},$$

which is Helmholtz's equation again. Writing the equation in full, with separate terms for the poles, we have

$$E_n + E_p = E_{nc} + E_{pc} + \theta \frac{dE_n}{d\theta} + \theta \frac{dE_p}{d\theta}.$$

The cell may be made up with the plates in separate vessels with a tube of electrolyte to connect them; and the vessels can be heated to different temperatures. As E_n, E_{nc}, and $\theta \frac{dE_n}{d\theta}$ depend on the temperature of their vessel, the last equation can be split up into two:—

$$E_n = E_{nc} + \theta \frac{dE_n}{d\theta} \text{ or } E_{nh} = \theta \frac{dE_n}{d\theta},$$

and

$$E_p = E_{pc} + \theta \frac{dE_p}{d\theta} \text{ or } E_{ph} = \theta \frac{dE_p}{d\theta};$$

and, as the temperature-coefficient of each contact can be found, the Peltier effect at each contact can be obtained separately.

The hypothetical cell discussed has only one fluid, and is reversible. It might be difficult to find such a cell. If two fluids are used, their junction can be arranged in a third intermediate vessel, and their Peltier effect, or difference of potential due to heat-formation, may be found as in the case of the plates.

Several workers, for instance M. Bouty and H. Gockel, have been investigating and measuring the temperature-coefficient or the Peltier effect directly.

Prof. J. Willard Gibbs has, of course, discussed the subject. In a letter to Dr. Lodge* he took a theoretical case in which this cell could be heated to the temperature of dissociation. He has written a second letter going more fully into the subject†.

This leads at once among the various conflicting views of electrolysis. One view, which seems tenable, is that electrolysis is always a case of double decomposition: that there is really a change. According to orthodox chemistry, if 2HCl is electrolysed H_2 and Cl_2 are obtained, and not 2H and 2Cl. The hydrogen and chlorine form combinations with themselves. Dissociation does not always split a compound in the same way as electrolysis, and the results of it are often "free atoms." It might be urged that in such a case as the electrolysis of NH_4Cl free atoms of NH_4 cannot exist, so NH_3 and HCl are produced by a sort of secondary action.

The cells discussed have been assumed to be reversible; and some question may arise as to what the chemical work means. For instance, in a Daniell cell are we to take such data as Zn,O=85,430 and Cu,O=37,160‡ from a convenient treatise on Thermochemistry, and to consider that the conversion into sulphate and other actions in the cell are secondary? It is to be regretted that writers on Thermochemistry seldom give their data so that their meaning is clear. Zn,O, for instance, often means that metallic zinc is burned in gaseous oxygen. Allowance must then be made for the physical states of the components before, and of the products after the combination.

---

* B. A. Report, 1886, p. 388.

† B. A. Report, 1888, p. 343.

‡ J. Thomsen, *Thermochemische Untersuchungen*, iii. pp. 275 and 320.

Returning to the reversible cell, however, before discharging a coulomb there are certain quantities of certain substances in certain physical and chemical conditions. After the discharge certain chemical and physical changes have taken place. Suppose any one of these is a secondary, or non-adjuvant action. For instance, suppose in discharging a Daniell cell that the conversion of zinc into oxide is a primary action, and the conversion of zinc oxide into zinc sulphate secondary or non-adjuvant. Some heat will be evolved, and the change from oxide to sulphate will not appear as external electrical work. On charging again there is only enough electromotive force at that contact to cope with the attraction of zinc for oxygen and not with the greater attraction for SO_4. The result is that the cell is not reversible. Any secondary or non-adjuvant action in the cell thus means non-reversibility. We thus have to deal not merely with Zn,SO_4 or ZnO,SO_3aq, and so on, but in a reversible cell, like a Daniell, we have to consider the attraction of Zn for SO_4; the physical change of Zn from a solid to a liquid state, which is generally included in the "heat of combination," as given; the solution of the $ZnSO_4$; and perhaps such small matters as change of volume of the cell. The copper salts must be dealt with similarly. Even if the heat of solution were non-adjuvant the cell would not be reversible.

Since 1883 I believe I have been alone in holding that not only is lead sulphate formed on both plates of a secondary battery, as shown by Dr. Gladstone and Mr. Tribe in 1882, but that its formation is the cause of the action of the cell, and not a secondary reaction at all. That is to say, there is no intermediate formation of PbO. To find out E_c of a cell we must open a sort of book. On one side put the heat of formation of every compound formed on discharge, and on the other of every compound broken up on discharge; the balance, including the physical changes, giving E_c. For instance, in a storage cell we credit the formation of $PbSO_4$ on the spongy plate, including, of course, the change of the SO_4 into the solid state, and on the peroxide plate we again credit $PbSO_4$. Two equivalents of H_2O are also credited. Two equivalents of $H_2SO_4 + Aq$ are debited, also two equivalents of H_2SO_4, and one equivalent of PbO_2, allowing for the O_2 becoming liquid.

It has been repeatedly urged that as the formation of $PbSO_4$ takes place on both plates it must cancel out. As in the case of a Daniell cell we have to deal with the difference between the heat of formation of two sulphates, it is assumed we must also deal with differences in a secondary battery. A little consideration will show, however, that the cases are different. In a discharging Daniell sulphate of zinc is formed and sulphate of copper is decomposed, so one goes on the credit side, the other on the debit. In a secondary battery sulphate is formed on both plates on discharge, and is therefore credited twice.

The doctrine of the Conservation of Energy also teaches us that electrolytic or nascent oxygen and hydrogen, which many chemists regard as such valuable reagents, do not exist. What is the evidence in favour of the existence of nascent hydrogen? If some such metal as magnesium, or sodium amalgam, is put in dilute acid, bubbles come off. Many other metals act similarly. If such a compound as persulphate of iron is put in, too, it is reduced to proto-sulphate. From frequently observing effervescence when reduction is effected, it is easy to assume the effervescence is the cause of the reduction. The theory is that the energy supplied first produces hydrogen. A powerful attraction has just been overcome, and has been satisfied neither by combination with more hydrogen nor otherwise. The hydrogen then seizes on the persalt of iron, and takes away some of the acid radical, forming free acid, which in its turn acts on the oxide of the metal. This action is secondary, and may be supposed to evolve heat. There is thus waste of heat, and the cell is irreversible. A better explanation would be, that the metal can dissolve if it either reduces the persalt or evolves hydrogen. The reduction of the persalt needs less energy, so that takes place. When there is no reducible salt available, hydrogen is evolved; and as it has to be expanded into the gaseous form, a good deal of work has to be done on it. Evolution of hydrogen and reduction of the salt are thus alternate, not consecutive results. Similarly in an engine—the steam either works the engine or comes out at the safety-valve; it does not begin to lift the safety-valve, and then change its mind and work the engine in a nascent state. It must be remembered that the term oxidize has come to denote many other things than adding oxygen. For instance, adding any electronegative radical is called oxidizing. If sulphur had been as common as oxygen, no doubt we should always talk of sulphurizing. The use of the term "oxidize" has also led to the tacit assumption that in electrolysis the water is electrolysed and the other results are secondary actions of electrolytic oxygen and hydrogen. If there were such a thing as nascent hydrogen, putting a depolarizer, such as nitric acid, round the carbon plate of a Bunsen cell would not increase its electromotive force; it would merely make it heat more on discharge.

So far the cells considered have been reversible. It does not follow that a cell is always reversible, but, if not, there is at least one non-adjuvant action. As a good example of non-reversibility, aluminium and its solutions may be taken. Aluminium does not dissolve in dilute nitric or sulphuric acid, yet it cannot be deposited electrically from any known solution. Aluminium and carbon in nitric acid give only a small fraction of a volt.

From the definition of electromotive force adopted in this paper, the "seat of the electromotive force" is in the cell and not between the positive and negative metals outside. Yet the electrometer behaves as if there were a contact electromotive force. Dr. Lodge has attempted to explain this in accordance with the "chemical theory." He argues that when, for instance, zinc is exposed to the air, the oxygen either begins to combine with it, or actually combines with it, the particles of oxygen giving up their charge to the zinc on combining. This produces a difference of potential which increases till the oxygen's attraction for the zinc is counterbalanced by it. There is thus an electromotive force set up. Air being an insulator, the circuit is not completed. If the tendency to combine with oxygen can produce an electric stress which prevents combination, actual combination must either charge the metal, if insulated, or produce a current, if a path is allowed. If a clean piece of sodium is put on an insulating stand it goes on oxidizing, the amount of oxide formed corresponding to an enormous number of coulombs. Where do they go to? If the metal charges electrostatically, it must soon be millions and millions of volts below the potential of the air, and must discharge disruptively. Moreover, the millions of volts are far more than equivalent to Na_2,O. Dr. Lodge assumes that a single element is an electrolyte, whereas a cell can only discharge by double decomposition. That is to say, to produce such an effect the metal must tear the oxygen from a combination, the other radical combining with another less electropositive metal, or removing an electronegative radical from it. It might be argued that zinc and copper plates in chlorine water will give a current, and chlorine is a single radical like oxygen. But immediately the plates are inserted the chlorine combines directly with both metals, without giving any current. A three-fluid battery is thus produced. The zinc is in a solution of zinc chloride, and the copper in copper chloride, and the intermediate liquid is chlorine water. The cell then discharges like a Daniell.

Though the oxygen form of the corrosion theory of contact electromotive force may not hold, it is quite possible that the Volta effect may be produced by thin films of water. Water is even more difficult to get rid of than oxygen, and might easily cause the electrometer readings. It must be remembered in connexion with this, that water must be present to enable even a combination of a metal with chlorine to take place. It is well known that dry chlorine will not attack a dry metal. Even sodium may be left in contact with chlorine.

Even if the Volta effect is due to the presence of traces of an electrolyte, such as water, and not to a non-electrolytic combination with free oxygen, the "seat of the electromotive force" still depends on a mere definition. The term "electromotive force" is continually used in two senses. It is sometimes used to denote the difference between the potentials of two points, and sometimes to denote the rate of fall of potential.

Maxwell gives it the latter definition formally, but frequently writes of the electromotive force when he means the line-integral of the electromotive force, that is to say the difference of potential. In connexion with cells, "electromotive force" is always used to mean difference of potential, so the stricter meaning, rate of fall of potential, may be disregarded just now. If we define the difference of potential between two points as proportional to the work done on a unit quantity of electricity moving from one point to the other, we have a clear statement of what we mean. But if we attempt to make the measurement, we find the unit quantity of electricity must have some carrier; and when we use an electrometer, we do not measure the work done on a unit quantity of electricity moving from one point to the other; we really measure the work done when a particular conductor is used as carrier. It is not necessary to consider any part of the circuit as the seat of the electromotive force. There is a circuit, and the potential is cyclic. Similar cases occur in other branches of electricity. For instance, a "unipolar" dynamo may be made up of a rotating magnet with a stationary circuit. If the resistance of the internal circuit is negligible in comparison with that of the external, the whole expenditure of power is in the external circuit, and the fall of potential over the external resistance is sensibly the whole potential or electromotive force of the machine. There is, however, no way of finding the seat of this electromotive force. If the rotating magnet is supposed to carry "lines of induction" round with it, they cut the external circuit and produce electromotive force there, so that becomes the seat of the electromotive force. If the lines of induction are taken as stationary, the rotating part of the circuit cuts them and becomes the seat of the electromotive force. The effects on the external resistance are the same in both cases. If it is a voltmeter the readings are the same, and they remain so if the lines of induction rotate faster or slower than the magnet, or in the opposite direction. If the voltmeter is replaced by an electrometer, it is still impossible to say where the seat of the electromotive force is. The readings would be the same whether the lines of induction were stationary, or revolved and cut the leads to the electrometer, or revolved and passed through the electrometer as if it were a zigzag gateway, without cutting the metallic part of the circuit anywhere. If the electrometer is removed and a proof-plate is used, by touching one terminal with it and measuring the work done when it moves to the other, the result is the same. If the lines are stationary, the proof-plate is charged by one terminal and repelled by it and attracted by the other. If the lines rotate, the proof-plate is urged forward by the lines of induction cutting it at right angles to its path. The reading is the same in both cases. The lines of induction are a mere convention, and there is no way of finding the seat of the electromotive force. All that can be said is that it is cyclic, and that the difference of potential of any two parts of the circuit of the same metal can be measured. In some cases the seat where power is spent can be determined, and in others it cannot.

Similarly in the cell, the seat of expenditure of power can sometimes be told, as when there is a resistance in circuit; sometimes it cannot, as when the cell works a unipolar, or, by extension, any other motor. The difference of potential between any two points of the same material can be measured, but all that can be said is that the electromotive force of the whole circuit is cyclic.

XVIII. *The Theory of Dissociation into Ions, and its Consequences.* *By* SPENCER UMFREVILLE PICKERING, *M.A., F.R.S.*[*]

THE supporters of the present physical theory of solution hold that the majority of salts, acids, and bases, when dissolved in a large excess of water, are entirely resolved into their component ions. That the facts of the case warrant such a conclusion I have already disputed (Phil. Mag. vol. xxix. p. 490, and 'Nature,' xlii. p. 626); but theoretical objections of a fundamental character can, I believe, be raised against this dissociation theory, and, though I have already alluded to these, they have not yet been fully discussed. Many of the questions which are asked in the following pages can no doubt be answered satisfactorily, but as far as I am aware this has not yet been done, and it is certainly incumbent on the supporters of the theory to give physicists and chemists a clear conception of what their theory involves.

---

[*] Read March 20, 1891.

When hydrochloric-acid gas, for instance, is dissolved in water, the molecules, which were intact to start with, become resolved into their ions, so that each of these acts as if it were a separate unit. This I believe is, according to the theory, the total and only change which occurs: the water remains in the same state in which it was to start with. The resolution of HCl into H and Cl atoms has been held of necessity to involve an *absorption* of heat, an absorption considerably in excess of that which we know occurs when it is resolved into hydrogen and chlorine molecules; and, whether the ions are identical with free atoms or not, we have the positive statement of Arrhenius, the originator of the present dissociation theory, that the resolution of a body such as hydrochloric acid into its ions absorbs heat*. If then this, which is the only change, *absorbs* heat, whence comes the 17,300 cal. which are, as a matter of fact, *evolved* during dissolution?

In the communication to which reference has been made, Arrhenius does not consider the thermal results of dissolution, and a subsequent consideration of these seems to have led some of the supporters of the theory to hold a view diametrically opposed to that just quoted. They now hold, I believe, that the decomposition of molecules into their ions *evolves* heat; that that heat, which they still admit must be absorbed by the decomposition of the molecules into ordinary atoms, is more than counterbalanced by the combination of the atoms with electric charges. This change of front must rather be inferred indirectly from the writings of dissociationists than from any definite retraction which they have published; nor does it appear to have been followed by all the supporters of the theory, for the explanation given by Arrhenius of the constancy of the heat evolved on neutralizing acids with bases is that it is in all cases due to the combination of the ions H and OH to form H_2O, and this explanation was quoted as recently as September last (B. A. meeting) by Shaw as being one of the strongest arguments in favour of the theory. It may also be remarked that up to July 1889 Ostwald seems to have held both views, and to have adopted either just as the exigencies of the case suggested: he explains the normal heat of neutralization as being due to the heat evolved in the formation of a molecule from its ions ('Outlines of General Chemistry,' 1890, p. 368), and the abnormal heat of neutralization as being due to heat evolved in the formation of ions from a molecule, though not, of course, the same molecule as in the previous case (p. 369).

---

* Arrhenius, *Bitrag till Kongl. Svenska Vetenskaps-Akademiens Handlingar*, Stockholm, 1883–4. Out of several sentences I may quote the following italicised passage:—" La transformation de l'état inactif en l'état actif d'un hydrate (faible) est accompagnée par une absorption de chaleur;" and the heat absorbed in the transformation is named "chaleur d'activité." The terms used in the above must be translated into the more modern language of the dissociationists, thus: inactive = undissociated; active = dissociated into ions; hydrate = any hydrogen compound, such as an acid or base; feeble = not much dissociated. See also Lodge's epitome of Arrhenius's paper, B. A. Report, 1886, pp. 361–384.

The first point, therefore, on which the dissociationists should give us definite information is, whether the dissociation of a molecule into ions is supposed to evolve or absorb heat.

Presupposing that the answer will be that heat is evolved (at any rate in cases similar to that of hydrochloric acid), their theory cannot be said to be *primâ facie* inconsistent with the conservation of energy; but other very serious difficulties arise which call for explanation.

The idea of heat being evolved by the combination of a charge with an atom involves the conception that the charge is originally independent of the atom: indeed the main idea of the theory seems to lie in the distinction between an ion, or charged atom, and an ordinary or uncharged atom*. We may ask, therefore, whence come these charges? All the ordinary means by which bodies become charged seem to be absent in the present case. No external energy has been expended, no friction can be supposed to exist except such as might result indirectly from an attraction between the water and the acid; but even if the existence of such an attraction were admitted, it could never cause sufficient friction to overcome the very attraction which is the original cause of it, to say nothing of the still stronger attraction which holds the atoms together: induction cannot apply, as both water and acid are supposed to be uncharged to start with, and even if there were a contact difference of potential between these two substances, it would not result in communicating both the + and — charges to one only of the bodies brought into contact,—the acid.

---

* *Cf.* Ostwald, *loc. cit.*, p. 275:—"What actually exists in the solution is single potassium atoms with enormous electrical charges. We do not know what those charges are in reality, but this we do know, that the chemical properties of substances are greatly altered by electrical charges. . . . As soon as the potassium atoms in solution lose their charge, as they do, for example, when led by an electric current to a platinum wire, where they can give up their electricity, potassium with its ordinary properties is at once produced, as is seen in its ability to decompose water with evolution of hydrogen."

In the second place, how can we imagine that an electric charge, which we must at present regard as an affection of matter, can combine with matter to produce heat and itself remain *in statu quo*? Such a view is little less than endowing the charges with some of the exclusive properties of matter, and calling this new matter into existence just when and where may be most convenient to the theory.

In the third place, how can it be maintained that the positive electrification of the hydrogen, and the negative electrification of the chlorine, would dissolve the union between them? According to all our experience of electricity, such electrification would make them cling together all the more firmly. Further, if these so-called + and — charges repel each other, why are they attracted by the — and + charges respectively on electrodes during electrolysis? or why, again, do the similarly charged atoms not attract each other (as dissimilarly charged ones are supposed to repel each other) and form hydrogen and chlorine molecules?

That a molecule, when decomposed by some force superior to the attraction of its constituent atoms, gives rise to free atoms which are possessed of a certain amount of free energy and that this free energy, which we call chemical affinity, may really be of the nature of an electric charge, has received the support of the greatest chemists and physicists whom Science has known; but the present theory seems to have nothing in common with such a view—indeed, it seems to be directly opposed to it. On the old theory the atoms when separated have *more* free energy than when combined, on the new theory they have *less*: on the old, the electric charges are the *consequence* of decomposition by some superior force, and form an integral part of the stuff resulting from the decomposition; on the new, they are the *cause* of this decomposition and are something outside and independent of the matter itself. The old theory attributes chemical affinity and combination to the existence of these charges; the new theory considers the charges to be antagonistic to chemical affinity, and to be the cause of chemical decomposition.

The view has been suggested, I believe, that the supposed dissociated atoms, though no longer held together by chemical attraction, may be still held together by the electrical attraction of their charges. This seems to be but an attempt to overcome a difficulty by changing a name, and so far from really diminishing the difficulty, it would appear only to increase it: for heat has been evolved, and, therefore, the state of combination is more intimate than it was before dissolution, so that the matter must be held more firmly together by these electrical charges than it was by its chemical affinity: how does this help the statement that they are *less* firmly united now—so much less firmly according to the theory, that they act as independent units? The difficulties as to the origin of the charges and the antagonism of chemical and electrical attraction are, moreover, not removed by this method of expressing the theory.

Another view, again, was suggested at the recent meeting of the British Association: that, instead of regarding the ions as atoms with electric charges, they might be regarded as allotropic modifications of the atoms themselves. This appears to me to be but hypothecating a new form of matter to satisfy a theory which is inconsistent with known matter, and, inasmuch as atoms of the same substance cannot differ from each other except by possessing different quantities of energy, it practically amounts to the conjuring away a stock of energy that the theory may not be said to be contradicted by the principle of the conservation of energy. But surely such a process is in reality as much a violation of this principle as writing $2=4$ would be. The energy equation will not equate, so the excess of energy on the one side is boldly struck off by imagining a new form of energyless atom, just as on the electric-charge theory the same is done by saying that the superabundant energy has been expended in combining with charges which have come from nowhere.

Whereas the potential energy of the ions of a substance such as hydrochloric acid must be regarded as less than that of the molecules when gaseous, it would appear that it must be greater than that of the molecules when solid—at any rate in such cases where the solid dissolves with an absorption of heat. Thus, in the case of potassium chloride, 74·5 grams of the salt dissolve in water at 0° with an absorption of 5184 cal. Let P and K represent the potential and kinetic energy of the solid (KCl) and of the ions (K + Cl) respectively, then

$$\mathrm{P(KCl)+K(KCl)=P(K+Cl)+K(K+Cl)-5284\ cal.}$$

The kinetic energy of the solid, K(KCl), is possibly an unknown quantity, but it is certainly a positive quantity; the kinetic energy of the dissolved substance, K(K + Cl), can, according to the osmotic-pressure theory, be calculated. With a monatomic gas no intramolecular work is possible, and therefore the pressure caused by that gas is a measure of its total kinetic energy (*cf.* Ostwald, *loc. cit.* p. 76); the pressure of each ion is in this case, *ex. hyp.*, the same as that which an ordinary molecule produces, and the pressure-producing energy of a gram-molecular proportion of such a gas is 34,008,000,000 ergs, or 819 cal., so that

$$\mathrm{K(K+Cl)=2\times 819\ cal.,}$$
$$\text{and } \mathrm{P(KCl)=P(K+Cl)-3646\ cal.-K(KCl).}$$

Thus the potential energy of the ions must be considerably greater than that of the solid molecule.

The same substance might have been taken as an instance of a gas dissolving with evolution of heat and a solid with absorption of heat; and in such a case the potential energy of the substance in solution must be intermediate between that of the gaseous and solid molecules. It is certainly difficult to imagine that this substance can consist of uncombined atoms.

But a still greater difficulty arises in some cases: a solution of calcium nitrate, for instance, absorbs heat on dilution, and, as the only change produced by diluting an already dilute solution is, according to the theory, to dissociate into ions some of the molecules still remaining intact, and as these molecules are present in the uncombined and gaseous condition, it follows that the dissociation of gaseous calcium nitrate molecules must absorb heat; therefore the gaseous salt on being dissolved in excess of water must absorb heat, the liquid salt would absorb more than the gas (by an amount equivalent to the heat of vaporization of this salt), and still more would be absorbed by the solid; yet direct experiment shows that this last, instead of giving a large absorption, actually *evolves* 4000 cal. when it is dissolved.

The conception that salts &c. in solution are entirely decomposed into ions has been regarded as a development of the theory of Clausius that a few free ions exist at any given moment in a mass of solution. But these two views appear to me to be radically different. Clausius's conception (Phil. Mag. 1858, vol. xv. p. 100) was of a two-fold nature. (1) That the molecules in any fluid, being at different temperatures at different times, owing to the impacts to which they have been subjected, some of them may occasionally be so hot as to be above their dissociation temperature, and that some temporarily free atoms would therefore be present. (2) That two similar molecules MR and M′R′ might collide under such circumstances that M′ might come nearer to R than M was, and, consequently, M′R would be the result; whether, however, M and R′ would find themselves in such close proximity that they would instantly combine together, or whether they might remain for a time actually free, is a question which our ignorance of the distances and forces concerned do not allow us to settle. Williamson's theory (Phil. Mag.) was practically identical with Clausius's second proposition—continual interchange of radicals, but not necessarily the presence of free atoms. Clausius's theory was proposed to account for the facts of Electrolysis, Williamson's to account for Chemical facts. Nothing which has come to light since that time seems to have shaken the idea of a continual interchange of atoms in the molecules of a liquid; there seem, however, to be decided objections to the conception of free atoms being present. Whether it is probable that in a liquid at ordinary temperatures there can be any molecules as hot as the dissociation temperature (which is probably 1000° to 2000°) or not, is a matter of opinion; but if any free atoms are present the chances of their meeting atoms of the same nature would be equal to those of their meeting atoms of the opposite nature: with the former they would combine just as they do with the latter, and the result would be that hydrogen and chlorine gas would be formed, and the acid would gradually become entirely decomposed: nor can it be argued that the similar charges on the free atoms of the same substance would prevent these combining, for in the analogous case of hydriodic acid we know as a fact that free iodine is produced when the acid is heated to its dissociation temperature.

It does not seem necessary, however, to imagine the presence of free atoms to explain the phenomena of electrolysis (*cf.* Lodge, B. A. Report, 1887, p. 338). The facts of the case, I believe, are that although an E.M.F. of finite magnitude is required to produce sensible electrolysis, *i. e.* the liberation of gas, &c. in recognizable quantities, any electromotive force will produce results indicative of electrolysis, these results being a gradual leakage of electricity, and a reverse or polarization current on removing the battery. The leakage may be explained by the electrolysis of those molecules which happen to be at a temperature above that of the average molecule, and decomposable by a lower E.M.F. than is the average molecule, or else by the action of the E.M.F. on those dissimilar atoms which at the moment when they lose their original partners find themselves comparatively very far apart from each other (M and R′ in the case above cited). As to the polarization current I venture, though with considerable diffidence, to ask whether it really is evidence of actual electrolysis. It is explained by the statement that a coating of liberated ions is formed on the electrode, and that these cannot be discharged so as to become ordinary atoms till the potential of the electrode attains a certain value, they consequently remain there and cause a reverse current. But might not such a current be caused by a similar coating of simple but undecomposed molecules; such molecules are, I believe, not fully saturated compounds, but still possess a certain amount of residual affinity, or, on the older electro-chemical theory, a certain amount of unneutralized charge; these would be attracted to the electrodes and would present their + and — ends to the — and + electrodes respectively: a charge would thus be retained on the electrodes after the battery was disconnected, and this charge would cause a reverse current when the two electrodes were connected together. The action in fact would be similar to that in a condenser. Such an explanation may obviate objections which can, perhaps, be urged against the idea of a coating of ions, for it seems difficult to see why a certain E.M.F. should be reached before the atoms can discharge themselves, unless we imagine a definite attraction between an atom and its charge, or why the E.M.F. required to effect this discharge should not always be the same whenever the same element is liberated; it also obviates the necessity of regarding an ion as possessing any form of charge which a free atom does not.

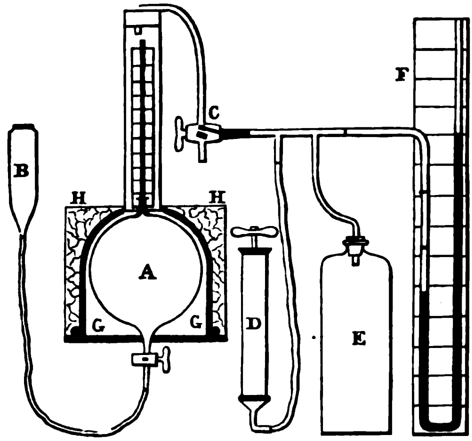
Whatever be the value of these suggestions, and of the objections raised against that part of Clausius's conception which supposes the presence of a *few* free atoms in a liquid owing to the accidental superheating of some of the molecules, it must be borne in mind that this view is totally distinct from the modern dissociation theory, that *all* the molecules are dissociated, and that too not by heat but by their affinity for electrical charges of inscrutable origin, and possessing hitherto unknown characteristics.

XIX. *An Apparatus for Measuring the Compressibility of Liquids.* *By* S. SKINNER, *M.A., Demonstrator at the Cavendish Laboratory, Cambridge*[*].

THE special features of the apparatus are (*a*) its very large bulb, (*b*) an arrangement to facilitate the filling and emptying processes. The volume of the globe is 1300 cubic centimetres, so that a small alteration of pressure, such as half an atmosphere, produces a considerable movement of the index in the capillary tube of which the capacity per cm. is ·003 cubic centim. To facilitate the introduction of liquids a second tube, closed by a stop-cock, is sealed in at the lower portion of the globe, and this is connected by a rubber-tube with a side reservoir. If a liquid be poured in the reservoir and the stop-cock be opened, the liquid will flow into the globe. The apparatus is supported on a base-board through which the tube carrying the stop-cock passes. A bell-jar stands on the base-board and is in connexion with a condensing air-pump and a mercurial pressure-gauge.

---

[*] Read May 9, 1891.



The behaviour of the apparatus has been tested by using it for a determination of the compressibility of water, with the following results :—

	Found.	Calculated from Tait's Extrapolation Formula.
At 5°·45 C.	·0000510	·0000502
9°·4	·0000501	·0000488
12°·4	·0000485	·0000479
16°·2	·0000480	·0000468

The apparatus has been applied to the comparison of the compressibility of solutions with that of the solvents, and the results exhibit the same general relations as with other properties of solutions. The solutions divide themselves into two classes, which may be broadly termed electrolytic and non-electrolytic. In the first there is a very considerable decrease in compressibility, as much as 8 per cent. with a 3-per-cent. solution of NaCl; whilst in the second there is but a slight diminution, quite beyond the limits of accuracy of the apparatus. For instance, a 5-per-cent. solution of naphthalene has a diminution of under 1 per cent.

A strict interpretation of the theory of osmotic pressure in solutions has been made by Prof. J. J. Thomson, in his "Applications of Dynamics to Physics and Chemistry," § 97, where he finds that 1 gram-equivalent per litre would decrease the compressibility by 1 part in a thousand. Non-electrolytes in dilute solution appear to follow this law; this is in agreement with their other properties, such as alteration in boiling- or freezing-points.

The diagram will serve to explain the general arrangement of the apparatus.

XX. *Note on* Kohlrausch's *Theory of Ionic Velocity.* *By* W. C. Dampier Whetham, *B.A., Coutts Trotter Student of Trinity College, Cambridge* [*].

Kohlrausch calculates his numbers for the specific velocities of different ions from measurements of the conductivities of salt-solutions, and of their migration constants, on the supposition that all the molecules of the salt present in solution are actively concerned in conveying the current. The values thus obtained were found to agree with experiment in certain cases by Prof. Oliver Lodge, and an investigation I am now engaged in carrying out seems also to confirm them. It seems to be generally supposed that this is inconsistent with any theory (such as that of dissociation) which supposes only a certain part of the salt to be active (see Lodge, B. A. Report, 1886, p. 391), though some such form of theory seems to be required by the relations shown to exist by Arrhenius, Van't Hoff, Ostwald, and others. If we examine the matter a little more closely, however, I think the two suppositions can be reconciled. Suppose that the ratio of the numbers of the active and the inactive molecules (which is generally supposed to measure the "dissociation") represents in reality the average ratio of the time during which each molecule is active to the time during which it is inactive. Every molecule is in turn active, but *at any instant* only a certain fraction of the molecules are active. [In terms of the dissociation hypothesis, the dissociation ratio measures the ratio of the mean free time to the mean paired time of the ions.] This is, of course, equivalent to supposing a certain fixed fraction of the whole number of molecules to be active, as far as statical effects, such as osmotic pressure, are concerned, but when we consider the velocities of the ions the case is different.

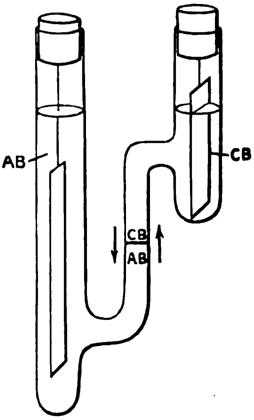
---

[*] Read May 9, 1891.

Kohlrausch calculates the relative velocity of the two ions $U=u+v$ from the molecular conductivity k/m, where $k=$ specific conductivity of the solution, and m its contents in gramme equivalents of salt, $U_1=u+v=k/m$. If now we suppose that at any instant only $1/n$th of the number of molecules are active, we should apparently have to put $U_2=\frac{kn}{m}$ *in order that the same current may be carried*, which would give $U_2=nU_1$.

But this U_2 represents the actual velocity of the ions while they are "free," and if we take a "dynamical" view of the dissociation equilibrium, they are only free for $1/n$th of their time; while combined they have no relative velocity, and so their *average* velocity for any *long* time is $\frac{1}{n}U_2=U_1$, the same as on Kohlrausch's hypothesis.

The investigation alluded to above, at which I am now working, seems to yield excellent results for certain cases, though it is of somewhat limited application. It consists in observing the phenomena at the junction of two salt-solutions, one of which is differently coloured to the other, when a current of electricity is passed across it. Salts are chosen which have one ion in common and the other different. Let us represent them by AB and CB, and consider the junction phenomena. The effect of the molecular interchanges will be a motion of B ions in one direction, and a motion of C ions and of A ions in the other. When a C ion crosses



the boundary, it again forms CB, but the colour of CB is different to that of AB, hence the boundary between the colours will move.

The method will be discussed when more experimental results are obtained; it appears that by measuring the rate of this motion the velocity of the ions can be arrived at. The present is merely a preliminary communication in explanation of the experiments shown to the Physical Society of London on the occasion of their recent visit to Cambridge.

XXI. *A Steam-Engine Indicator for High Speeds.*

By Prof. JOHN PERRY, *F.R.S.*[*]

MEMBERS who are not practically acquainted with the errors of the ordinary steam- or gas-engine indicator are referred to a paper in the 'Proceedings of the Institution of Civil Engineers' (vol. lxxxiii. 1885) by Professor Osborne Reynolds, and an exhaustive discussion of the subject.

Whether we do or do not share Prof. Reynolds's view that even at low speeds of engines the errors in calculating horse-power are very considerable, because of the friction of the paper barrel and the stretching of the cord or wire which gives to the paper barrel a miniature motion of the engine piston, it is obvious that there will be more confidence in an indicator in which the motion-copying mechanism gives a very short stroke, is opposed by very small forces, and requires no string or wire. I do not pretend to get rid of the error in a spring which is due to change of temperature, but where great accuracy is required it is always worth while, when using this or any other indicator, to have a "boiler-pressure" line drawn upon the diagram, as well as an "atmospheric line;" and these two lines, with a pressure-gauge on the boiler, will give the scale to which the indicator at the time represents pressure and prevent error due to change of temperature of the spring.

In these days of high-speed machinery the most important defect of an indicator is its slow natural period of vibration.

---

[*] Read May 22, 1891.

If the natural period of vibration is from $\frac{1}{20}$ to $\frac{1}{30}$ of the time of revolution of an engine, it is found that the diagram is not deformed by waviness, ordinary fluid friction destroying the natural vibration; if it is as much as $\frac{1}{15}$, a considerable amount of pencil pressure must be employed to obtain a frictional stilling of vibration; but if it is as low as $\frac{1}{10}$ it is almost impossible, even with great pencil friction, to obtain a decent diagram. To make an indicator have a quicker natural vibration it is necessary to use a stiffer spring, and this means that pressures are indicated to a very small scale. Introducing seven per cent. of inaccuracy in area of diagram, either by friction or smallness of scale, may possibly allow the very best existing indicator to be used on engines of as high a speed as 400 revolutions per minute; but this is a matter on which assertions are made of higher and very much lower limits of speed than what I have stated. I think that users of indicators will generally agree that I have not exaggerated this defect; but unfortunately there is scarcely ever any kind of check which can be applied to a measurement of indicated horse-power, so that mere assertions are of very little value.

Indicated horse-power is now the sole standard of the values of engines, so that its accurate measurement is very important; but for the inventor and improver of engines, and for the physicist, that there shall be no local deformations in the shape of the indicator-diagram is of greater importance than that its area should be correct. Now the natural period of vibration of the indicator before you is about $\frac{1}{500}$ of a second; allowing twenty periods to one revolution of the engine, I find that this indicator will give diagrams with no wave-deformations until the speed of the engine exceeds 1500 revolutions per minute. I can, however, by changing the disk make the natural period of vibration $\frac{1}{1000}$ or $\frac{1}{250}$ of a second, or, indeed, what I please, so that there is practically no speed at which this indicator will not give a true diagram.

Specimens of the indicator are before you, but I will throw upon the screen drawings of it. Fig. 1 is a section, and fig. 2 an elevation. It consists of a very shallow circular box, E, of metal (in some of my specimens it is of cast iron, in others of gun-metal) closed by a thin disk of steel, D.

Here, for example, is one an inch and a quarter in diameter, and about $\frac{1}{60}$ of an inch thick, which I use for maximum steam-pressures of about 30 lb. to the square inch above the atmosphere. It was my intention to use corrugated disks, but as these could not be obtained cheaply, except in quantity, I have hitherto used plane disks, and found them quite satisfactory for such sizes of diagram as I have hitherto dealt with. I am, however, making arrangements for obtaining corrugated disks for use in the indicator.

Fig. 1.

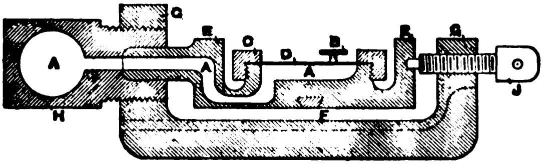
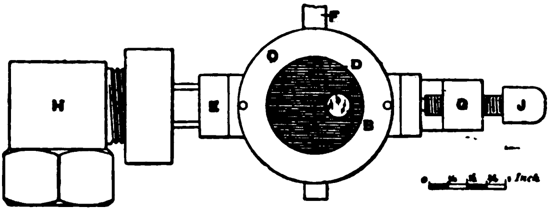


Fig. 2.



Now when this box E is put in communication with the cylinder of a steam-engine in the usual way, that is through the pipe A and the indicator-cock, the disk yields more or less as the pressure is greater or less. To magnify this yielding I fix upon the disk, about halfway between its centre and circumference, a small mirror B such as is used in electrical laboratories. This mirror has a light frame of metal, and can be screwed to or unscrewed from any of these disks quite readily. I let a beam of light from an ordinary oil-lamp fall upon the mirror, and it is reflected and falls upon a sheet of white paper which it illuminates at a small spot. Now the yielding of the disk under fluid pressure is evidenced by the movement of the spot of light on the paper. For example, if steam of 10, 20, or 30 lb. pressure per square inch (above the atmosphere) is admitted to this box with its present disk, and the spot of light is on a sheet of paper about 4 feet away from the mirror, the spot will be seen to travel one, two, or three inches from its original position. If there is a partial vacuum inside the box, the spot travels in the opposite direction. It will be seen, therefore, that I have used a reflected beam of light as if it were a rigid pointer four feet long. By using a lens, by using magnesium light instead of a common oil-lamp, and by taking certain precautions which are quite obvious to the members of this Society, I could throw a well-defined spot upon a screen forty feet away from a mirror upon a small corrugated disk, whose motion would be with great exactness proportional to the pressure, for motions of as much as five or six feet.

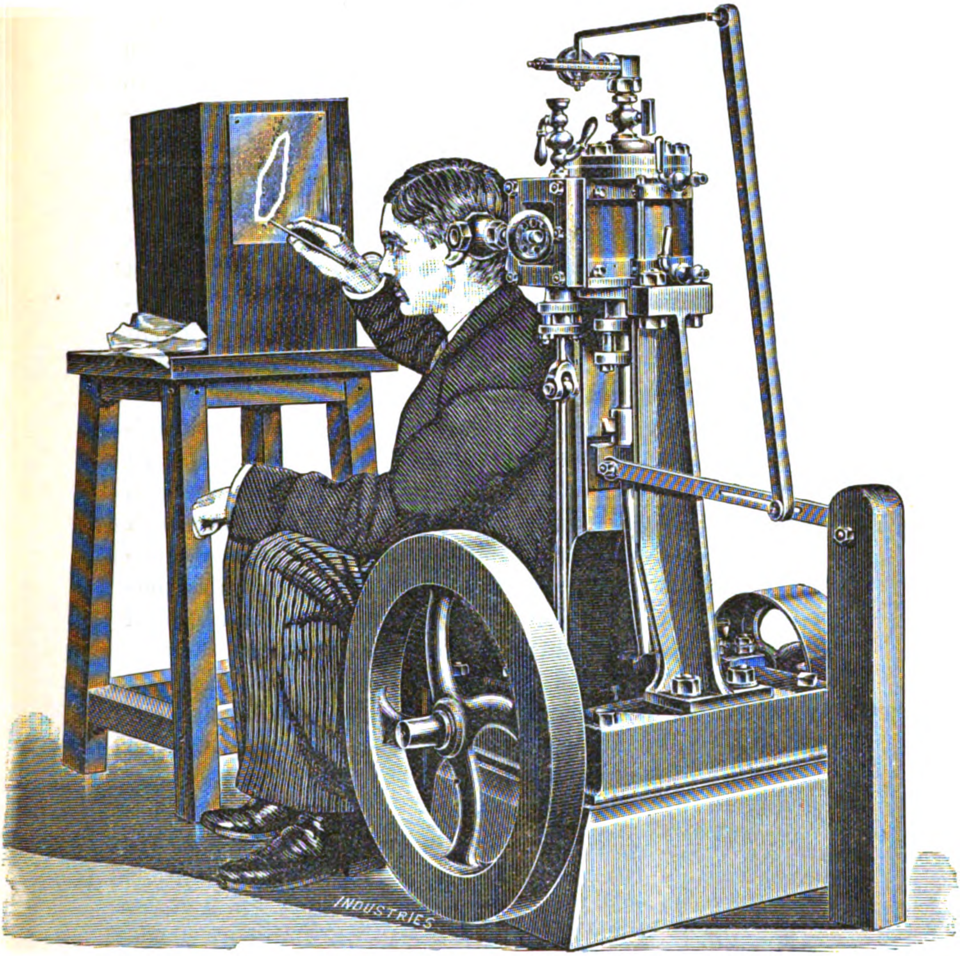
The end of the arm F receives a miniature motion of the piston of the engine by stiff rods, and this causes the spot to move at right angles to its former motion; and when both motions are being given the spot travels round on the screen, its position at any instant indicating the pressure of the steam and the position of the piston in its stroke.

Now, although I have never heard of such a method being used, I feel that many people must have thought of using it. I myself thought of it many years ago; but the making of a photographic record seemed to me to introduce great difficulty, so I never tried it*. What I have now discovered is this, that a photographic method of recording is quite unnecessary. In fact, even at speeds of 60 revolutions per minute the image of the spot remains on the retina sufficiently long to enable a man to draw upon the screen the path of the spot. He first turns the indicator-cock, so that there is atmospheric pressure under the disk. The spot now travels in a straight line, and this is the atmospheric line. When I wish to check the scale I do what I should like to see done with all indicators, I let steam at boiler-pressure underneath the disk and mark out another straight line parallel to the atmospheric line. I now let the box communicate with the cylinder, and I draw the actual diagram. A very little practice will enable anyone to draw the diagram quite accurately, even when the engine makes only 60 revolutions per minute. But at such speeds as 150 revolutions per minute, the diagram is quite continuous as a thin line of light on the paper, and the most unskilled person need not make errors of as much as one per cent. in drawing a pencil line, which remains quite visible in the middle of the thin line of illumination.

---

* Prof. R. H. Smith has, since the reading of the paper, called my attention to an indicator described in 'Engineering' of July 10th, 1885, by Messrs. Clarke and Low, in which a reflected beam of light is used for magnification.

Fig. 3.



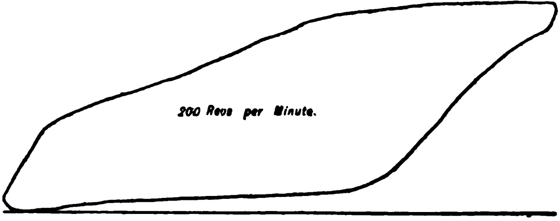
Using a common oil-lamp, a diagram about six inches long and four inches broad, formed of a band of light one tenth of an inch broad, is quite visible even in a well illuminated room. If the room is darkened the diagram becomes quite vivid, and to anyone accustomed to indicator-diagrams it is an object of interest from quite a number of considerations.

As I am unable to show to the Society this instrument in action, I will throw upon the screen a photograph, fig. 3, taken at Finsbury of the indicator placed upon a toy steam-engine, which some of my students made to drive testing machines. My partner, Mr. Holland, is following the spot of light with a pencil. As a matter of fact, the engine was really at rest, and the diagram was drawn as a thick chalk line upon dark paper, when this photograph was taken; but if an instantaneous photograph had been taken of the arrangement actually working, it would not have been very different from what I here show, except that the line of light forming the diagram would have been finer.

We have only been able to run this toy engine at a maximum speed of about 900 revolutions per minute, and indeed at this speed it was rather dangerous to run it, as, through want of balance, it set the floor and everything in the room a-shaking; but even at this high speed there was no evidence on the diagram of any waviness due to the natural vibrations of the disk of the indicator. There was, however, a musical note in evidence of the existence of such vibrations.

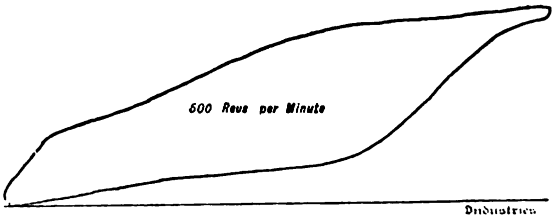
Mr. Holland has used a tapering wooden box as if it were a camera for photographing the diagram and its atmospheric line, and he tells me that it is quite easy to take

Fig. 4.



such photographs. He is, however, accustomed to such work, and I do not think that an ordinary engine-driver would care to try the photographic method. I here exhibit some of Mr. Holland's photographs. This one, fig. 4, required an exposure of one minute, the light being that of an ordinary oil-lamp. This one, fig. 5, required an exposure of ten seconds, the light being obtained by the burning of a piece of magnesium strip behind the hole.

Fig. 5.



I have, however, used Mr. Holland's box with sheets of tracing-paper instead of photographic plates, and the indicator-diagram is now an object which is very easily observed and traced.

I exhibit a few drawings made on tracing-paper in this way by Mr. Holland. First, we have a few taken at about the same time; these taught us that we had better make considerable changes in the valve of our engine. We made these changes, and here are some others taken subsequently. The speed and other necessary information are given on each sheet of paper, so that the effect of change of speed on our toy engine may be noted.

With an ordinary indicator the instrument must be stopped in its action, and a sheet of paper taken from the drum, before the diagram can be looked at. But here the diagram is visible all the time. To the student or improver of steam-engines it is very instructive to keep looking at the diagram whilst altering steam-pressure, or speed, or load of the engine, and it is an amusement of which one does not very soon become tired. Even with this toy engine of mine I have already observed such changes in the shape of the diagram as have thrown a perfectly new light upon the condensation phenomena occurring inside a steam-engine cylinder.

The first form of this new indicator is lying on the table. You will see that to give it the piston motion in miniature required considerable force to be exerted, a fault which I have corrected since. Now the very first time the instrument was tried a very curious phenomenon was observed, namely, that the indicator-diagram was not one continuous line of light, but a series of "blobs" or spots, connected by a much fainter line. I easily saw that these indicated a vibration going on in the rods which gave to the box the motion of the piston in miniature. I found that the cause of this was due to the bad fitting or "backlash" in a lever which my students had been in the habit of using with ordinary indicators on the same engine. With an ordinary indicator there was no possibility of observing from the diagram that such a fault existed, and was probably greatly accentuated by the use of a cord instead of our rod; here it was very evident. Indeed, although only to a small extent, it will be observed in the photographs that there are regular changes of intensity of light in every diagram, and that this vibration of the piston-motion mechanism has not been altogether done away with even in our more carefully fitted stiff rods.

Added, June 5th, 1891.

In the discussion which followed the reading of the paper it was suggested by Mr. Swinburne that the most useful function of such an indicator would be to continually show the diagram on a screen in an engine-room, so that at every instant the engineer would not only know the pressure of steam in his boiler, but also the pressure at every instant in the cylinder.

I quite forgot to put this in my paper, although its importance was not unknown to me. I am now attaching an indicator to each end of the cylinder of my driving-engine at Finsbury. These will receive their miniature piston-motion from one rod. One gas-jet will throw from the two mirrors two diagrams upon a screen some seven feet away from the mirrors, and these diagrams will be visible from nearly every part of the engine-room at all times. In the room at present there are not only very visible pressure- and vacuum-gauges and a speed-indicator (not a counter), but also a dynamometer coupling, which shows at a glance the actual horse-power which is being transmitted along a shaft. If, instead of tracing the diagram with a pencil, I were to trace it with the point of a planimeter, the reading might represent the horse-power.

It is unnecessary to describe here how the indicator might be fixed to the valve-chest or receivers, or to other parts of an engine, such as the air- and force-pumps; or to pumping-machinery generally; or to engines driven by fluids. In fact it may be said that I have made the discovery that the beautiful method of observation of vibration of M. Lissajous has hitherto been neglected in its applicability to all sorts of practical purposes. Quite a number of other applications of the reflected beam of light principle suggest themselves.

XXII. *On the Value of some Mercury Resistance Standards.* *By* R. T. Glazebrook, *M.A.*, *F.R.S.*, *Fellow of Trinity College, Cambridge* *.

In a paper read before the Physical Society † on May 23, 1885, I described the results of a comparison between the original standards of the British Association and some copies of the mercury unit representing Legal Ohms made by M. Benoit in Paris and sent to me by him. Two of these copies have remained in my possession since that time; the third, at M. Benoit's request, was handed to Mr. Preece. The mercury with which the tubes were filled in 1885 has remained in them since that date. Now recent observations had seemed to indicate that a small change had taken place in some of the platinum-silver standards, and it became desirable to compare them again with M. Benoit's tubes. The method was the same as that of my former paper. The tubes immersed in melting ice were compared with the B.A. standards. The temperature of the room was kept very low, from 0°·5 to 3° C., and thus the errors caused by the conduction of heat into the tubes through the copper connecting rods were avoided. The connecting pieces used to connect the tubes to the bridge differed from those employed in 1885, being a modification of the platinum cups described in my paper* on the specific resistance of mercury. A hollow platinum cup, about 3·5 cm. long and rather more than 1 cm. in diameter, is secured firmly into an ebonite tube; the outside of the tubes are turned to fit the ground portion of the glass vessels which form the ends of the mercury tubes, thus taking the place of the stoppers which usually close the tube; the platinum cups dip into the mercury, the surface of contact being about 12 sq. cm. The cups had previously been platinized; on the present occasion they were merely cleaned with nitric acid and distilled water. Stout copper rods, well amalgamated, fit tightly into the inside of the cups; the other ends of these rods are in contact with the bridge. Thus the connexion with the bridge is made through the rods and across the platinum of the cups, avoiding contact between the mercury and copper.

---

* Read May 9, 1891.

† Glazebrook, Phil. Mag. October 1885; Proc. Physical Society, vol. vii.

The resistance of the connexions was determined, and it was shown that the resistance of the platinum was negligible. This was done by making the rod and cup part of a circuit, the resistance of which was measured. The rod was then removed from the cup, and the latter was removed, so that the part of the copper which had been in the cup was now in direct contact with the mercury into which previously the cup had dipped. No change in the resistance of the circuit was produced by this. To secure this result it was necessary that a considerable area of the platinum should dip into the mercury; and this condition was in the experiments always carefully attended to. The resistance of these connecting-pieces was thus found to be ·00291 B.A. unit.

The tubes were compared both with the B.A. units and also with the Legal-Ohm standards. In the comparison with the B.A. units, a large amount—about 300 divisions, or ·015 B.A. unit—of the bridge-wire was used. Since the resistance of the bridge-wire is known at 15°, this required, when the temperature of the room was as low as 1° or 2°, correcting for temperature; but as the whole correction amounted to less than 2 per cent. of the bridge-wire used, while this again was only 1·5 per cent. of the resistance measured, no very accurate knowledge of the temperature of the wire was needed. In some cases a coil of 100 B.A. units was put in multiple arc with the mercury tube; and the resistance of the combination, which amounted to a little over 1 B.A. unit, was compared with the wire standard.

---

* Phil. Trans. 1888.

The results of the comparisons are given in the following Tables.

Table I.—Tube, Benoit No. 37.

Date.	Standard.	Notes.	Value in B.A. units.
Jan. 3......	Flat.	....	1·01111
„ 5......	Flat.	....	1·01103
„ 5......	Flat.	100 B.A. units in multiple arc with tube.	1·01108
„ 6......	Flat.	....	1·01099
„ 6......	Flat.	Mercury drawn through.	1·01101
„ 7......	Flat.	....	1·01109
„ 7......	Flat.	Mercury drawn through.	1·01109
„ 7......	Flat.	After interval of 1 hour.	1·01106

Mean for No. 37...... 1·01106 B.A. units.

Table II.—Tube, Benoit No. 39.

Date.	Standard.	Notes.	Value in B.A. units.
Jan. 3......	Flat.	....	1·01053
„ 5......	Flat.	....	1·01058
„ 5......	Flat.	100 B.A. units in multiple arc with tube.	1·01060
„ 6......	Flat.	....	1·01049
„ 6......	Flat.	Mercury drawn through.	1·01037
„ 7......	Flat.	....	1·01041
„ 7......	Flat.	Mercury drawn through.	1·01034
„ 7......	Flat.	After interval of 1 hour.	1·01029
„ 8......	Flat.	....	1·01022
„ 8......	Flat.	Mercury drawn through.	1·01027
„ 8......	Flat.	After interval of 1 hour.	1·01033

Mean of last seven for No. 39...... 1·01032.

With regard to the observations on No. 37, it will be seen that they are extremely close, the greatest difference between any two is ·00012 B.A. unit, or rather more than 1 in 10,000; and no appreciable change was produced by passing the mercury through the tubes. The observations on No. 39 are not quite so close. On January 6 a distinct change was caused by passing the mercury through; after that date the results are fairly consistent, and no further alteration was observed; the mean of the values found after this date may be taken as the resistance of the tube to an accuracy of about 1 in 10,000.

The two results therefore are:—

No. 37...............	1·01106 B.A. units.
No. 39...............	1·01032 B.A. units.

The observations in 1885 were reduced to Legal Ohms, using the value for the resistance of mercury in terms of the B.A. unit adopted by the B.A. Committee in that year, and based on Lord Rayleigh's experiments. According to this 1 Legal Ohm = 1·0112 B.A. units.

If we take these values we have as the resistances in Legal Ohms the following:—

	Benoit.	R. T. G. 1885.	R. T. G. 1891.
37	1·00045	·99990	·99986
39	·99954	·99917	·99913

Comparing the last two columns we see that the tubes have apparently fallen in value relative to the standard Flat in the 5½ years by ·00004, but this quantity is too small to be really certain of. The results of the experiments therefore prove within this limit that the platinum-silver coil Flat has not altered relative to the mercury-tubes in this interval. The difference between the values found by Benoit and myself depends on the value used for the resistance of mercury in B.A. units. According to Lord Rayleigh *, the resistance of a column of mercury 100 cm. long, 1 sq. mm. in section, at 0° C., is ·95412 B.A. unit, and this is the value which has been used in the above. The value found by Mr. Fitzpatrick † and myself for this resistance was ·95352. Practically the same number has been obtained by Wuilleumeier ‡, Hutchinson §, and Salvioni ‖. If this number be adopted, then

$$1 \text{ Legal Ohm} = 106 \times \cdot 95352 \text{ B.A. unit.}$$
$$= 1\cdot 01073 \text{ B.A. units.}$$

---

* Phil. Trans. 1883.

† Phil. Trans. A. 1888.

‡ Wuilleumeier, *Comptes Rendus*, cvi., 1888.

§ Hutchinson, Johns Hopkins University Circulars, 1890.

‖ Salvioni, *Rend. della R. Acad. del Lincei*, vol. v. fasc. 7.

And then we find as the values of the tubes in Legal Ohms:

	Benoit.	R. T. G. 1891.
37	1·00045	1·00033
39	·99954	·99959

and these numbers show an extremely close agreement, thus proving that the resistance of a column of mercury 100 cm. in length, 1 sq. mm. in section, as given in this indirect way by Benoit's experiments, is very close to the value ·9535 B.A. unit.

In my previous paper * are given some values for the change of resistance of mercury with temperature. On January 3 I observed the resistances of the tubes Nos. 37 and 39 at a temperature of 12°·9 C. The results were:—

	Resistance at 12°·9.
37..................	1·02254
39..................	1·02193

From these we get as the values of the coefficients per 1° C., ·000875 and ·000870 respectively. Thus the mean coefficient between 0° and 13° is ·000872. This agrees well with the values found in 1885, the mean value between 0° and 10° being ·000861, and between 0° and 15° ·000879. From these we have as the value between 0° and 13°, ·000872.

Benoit † and Strecker ‡ give higher values than the above, Strecker's value between 0° and 13° being ·000906.

In the paper on the specific resistance of mercury, Phil. Trans. A, 1888, I have called attention to some of the consequences of this difference. The value of the coefficient between 0° and 10° found by Mr. Fitzpatrick and myself in 1888 was ·000876. Kohlrausch § and Strecker ‖ find as the values for the resistance of mercury in B.A. units, ·95338 and ·95334 ; these are less than the value found by Salvioni, Hutchinson, Wuilleumeier, and myself, and the difference is, I believe, mainly due to the uncertainty in the temperature coefficient. They both worked with their tubes in a bath at

---

* Phil. Mag. October 1885.

† Benoit, *Journal de Physique*, 1884.

‡ Strecker, Wied. *Annalen*, vol. xxv. p. 475.

§ Kohlrausch, *Abhand. der K. bayer. Akad. der Wiss.* ii. Cl. xvi. Bd. iii. Abt.

‖ Strecker, Wied. *Annalen*, vol. xxv. p. 475.

about the temperature of the room, and reduced their results to 0° by the use of Strecker's coefficient. The mean temperature adopted by Kohlrausch was 10°, he puts

$$\sigma_0 = \frac{\sigma_{10}}{1 \cdot 00904},$$

σ_0 and σ_{10} being the resistances at 0° and 10°. According to my own results in 1888 we should have

$$\sigma_0 = \frac{\sigma_{10}}{1 \cdot 00876};$$

and the value of σ_0 would be increased from ·95334 to about ·95360, which is rather greater than my value. The value ·000876 is given by Benoit for the mean coefficient between 0° and 10°.

XXIII. Mr. Blakesley's *Method of Measuring Power in Transformers.* *By* Prof. J. Perry, *F.R.S.*[*]

Mr. Blakesley's method of measuring the power given to the primary coil of a transformer becomes more important the more it is studied. Mr. Blakesley proved it to be correct if currents followed the simplest periodic law; if there was no magnetic leakage; if magnetic permeability was constant. Any person who has used Fourier's theorem knows that if Mr. Blakesley's rule is right for a sine function, it must be right for any periodic function whatsoever; as any periodic function may be expressed in sine functions, and each of these enters into the equations as if it were alone[†].

---

[*] Read May 22, 1891.

[†] This assertion was challenged in the discussion. Perhaps I ought to have explained myself more fully. At the time I happened to be working with Fourier's Series very much, and I lost sight of the fact that what was very evident to me might not be evident to others.

If $$x = \sum_1^\infty (a_i \sin ikt + b_i \cos ikt),$$

and $$y = \sum_1^\infty (\alpha_i \sin ikt + \beta_i \cos ikt),$$

where $k = \frac{2\pi}{\tau}$ and τ is the periodic time, then the average value of xy

Prof. Ayrton and Mr. Taylor have proved the method to be correct for currents of any periodic law, the permeability varying in any way whatever. But they make the assumption that there is no magnetic leakage.

I believe that it was Dr. Hopkinson who, in his paper read before the Royal Society on March 10th, 1887, first departed from the old-fashioned way of treating mathematically the equations concerning currents in neighbouring coils, and he has been followed by everybody else who has written upon that subject since. Some writers of eminence have given, and incompletely, Hopkinson's investigation, evidently not having seen his paper. In my opinion the usual method is somewhat misleading. Assuming no eddy currents in the conducting part of a transformer, the equations written in the old-fashioned, and in what I venture to say is the only correct way, become

$$\left.\begin{array}{l} V = RC + L\dot{C} + M\dot{C}' \\ 0 = R'C' + M\dot{C} + L'\dot{C}' \end{array}\right\} \quad \cdots\cdots \quad (1)$$

Here V is the voltage at the terminals of the primary circuit, R its resistance, C its current, and L its coefficient of self-induction. R′ is the resistance of the whole secondary circuit, in which we assume no independent electromotive force; C′ is its current, L′ is its coefficient of self-induction, and M is the mutual induction between the two circuits. It may be well to state that, using amperes, volts, and ohms:— If P and S are the numbers of windings of the primary and secondary respectively; if a is the cross section of the iron

---

between the limits 0 and τ is

$$\tfrac{1}{2}\Sigma(a_i \alpha_i + b_i \beta_i),$$

and does not involve any term such as $a \ \alpha_r$ or $b_i \beta_r$; that is, into the expression for the average value each term of the Fourier's Series enters just as if there were no other terms. Nearly all practical Electrical Engineers are in the habit of ignoring calculations which assume that a current is a sine function of the time; they say that such calculations are useless because the current never is a true sine function of the time. I have here given one of many examples which might be given in which a proposition concerning any periodic function need only be proved for one of the Fourier terms of that function. And in all cases, the result of the study of a sine function is at once applicable to any periodic function whatsoever.

in square centimetres, λ the average length of the complete iron magnetic circuit, and μ the permeability (being about 1500 in ordinary transformer working), we may take it that

$$L = P^2 \frac{a\mu}{\lambda} \frac{4\pi}{10},$$

$$L' = S^2 \frac{a\mu}{\lambda} \frac{4\pi}{10};$$

and if there were no magnetic leakage—that is, if all the field due to a primary current through every single winding of the primary passed through every single winding of the secondary, then $M = \sqrt{LL'}$, or

$$M = PS \frac{a\mu}{\lambda} \frac{4\pi}{10}.$$

But there is always some magnetic leakage, and it fills me with astonishment that so many investigators should assume that a little leakage makes no difference.

To get an idea of the importance of even a little leakage let us eliminate $\dot{C}_1$ from equations (1) and we have the result

$$V = RC - R' \frac{M}{L'} C' + \frac{LL' - M^2}{L'} \dot{C}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The usual assumption that if LL' only differs by a very little from M^2 the error is unimportant, is seen to be inadmissible when we consider how great a value $\dot{C}$ sometimes may have in comparison with C or C'. Thus, for example, in a transformer with which I have had something to do experimentally, $L = 15$, $L' = 0{\cdot}15$, $R = 10$, and M is very nearly $1{\cdot}5$; so that (2) becomes

$$V = 10C + 10R'C' + \left(15 - \frac{M^2}{L'}\right)\dot{C}.$$

Now, to take the very simplest kind of periodic current, and the one for which the above wrong assumption is least wrong, and a frequency of 106 per second—writing, in fact,

$$C = A \sin 1000\,t,$$

we know that

$$\dot{C} = 1000\,A \cos 1000\,t.$$

So that, even if M differs only by 1 per cent. from what it has been assumed to be, that is, if there is only 1 per cent. of magnetic leakage, the neglected term $\left(15 - \frac{M^2}{L'}\right)\dot{C}$ becomes of the value

$$\left\{15 - \frac{(1{\cdot}485)^2}{{\cdot}15}\right\}\dot{C},$$

or $0{\cdot}3\,\dot{C}$, or $300A \cos 1000\,t$. In fact, the neglected term becomes thirty times as important as the important and certainly hitherto non-neglected term RC in the equation.

Now in no case is the current truly a sine function of the time, and any departure from this simplest kind of periodic current makes the error of which I speak much greater.*

---

* As an example, one of many worked out by my students at Finsbury during the last few years: taking the sizes of iron from a certain Mordey transformer which I have occasionally used; assuming permeability constant and no eddy currents in copper or iron; assuming currents to be true sine functions of the time. If V is voltage at terminals of primary, primary resistance 10 ohms, internal secondary resistance 0·1 ohm, outside resistance of secondary in ohms being called ρ; self-induction of primary 15 secohms, self-induction of secondary 0·15 secohm; assuming V in volts $= 1000 \sin \frac{2\pi}{\tau} t$, and taking frequency 106 or $\tau = 1 \div 106$ second, it is quite easy to calculate to any number of places of decimals that may be desired, the amplitudes and lags of the primary and secondary currents, and indeed all other magnitudes involved. The graphic method of working is evidently quite out of the question.

My students have for several years made calculations of this kind, obtaining tables of values for various frequencies and amounts of iron in the transformer, and they are exceedingly instructive. Until such tables are compared with actual experimental results, it seems to me that debates as to the effect of hysteresis consist merely of assertions having no physical basis.

For my present purpose I will give part of two tables calculated by Mr. Eliott, one of my students. Taking the above values:—

1st. If we assume that there is no magnetic leakage. In that case $M = \sqrt{LL'} = 1{\cdot}5$ secohms. Using this value of M we get Table I.

2nd. If we assume that there is a little magnetic leakage, say one and one third per cent., or that $M = 1{\cdot}48$ secohms. Using this value of M we get Table II.

Now it is perfectly certain that there is some magnetic leakage, always; but it is rather difficult to say just how much there may be. I have here assumed in taking $M = 1{\cdot}48$ instead of 1·50 that $1\frac{1}{3}$ per cent. of the total induction due to the primary coils does not pass through the

Of course, any self-induction in the outside part of the secondary circuit will produce the same effect as a leakage in the transformer itself.

The interesting fact to which I wish to draw the attention of members of the Society is this, that however great may be the magnetic leakage, Mr. Blakesley's method is still correct if magnetic permeability is assumed constant during a cycle. Multiplying equation (2) by C we have

$$VC = RC^2 - r\frac{M}{L'}CC' + \frac{LL' - M^2}{L'}C\dot{C}. \quad . \quad . \quad . \quad (3)$$

Integrating for the whole periodic time and dividing by this time—that is, taking the average value of every term in (3) it is to be observed that

$$\frac{1}{\tau}\int_t^{\tau+t} C\dot{C} \, . \, dt = \frac{1}{\tau}\int_{C_0}^{C_0} C \, . \, dC = 0,$$

if C_0 is the value of C at the beginning and end of the period; and hence, as $\frac{M}{L'} = \frac{P}{S}$ very nearly,

$$\text{Average } VC = \text{average } RC^2 - \text{average } r\frac{P}{S}CC_1.$$

If either C or C' were the current in a non-inductive

---

secondary coils, and that $1\frac{1}{3}$ per cent. of the total induction due to the secondary coils does not pass through the primary coils. This number has been taken at random.

The meanings of the letters used at the heads of the various columns are these :—

If
$$V = 1000 \sin\frac{2\pi}{\tau}t,$$
$$C = A\sin\left(\frac{2\pi}{\tau}t - \epsilon\right),$$
$$C' = A'\sin\left(\frac{2\pi}{\tau}t - \epsilon'\right),$$
$$V' = a'\sin\left(\frac{2\pi}{\tau}t - \epsilon'\right) = \rho C',$$

P = average power given to primary,
P' = average power given out by secondary.
Percentage efficiency = 100P'/P.

Evidently V' is the voltage at the terminals of the secondary circuit. Angles of lag are given in degrees.

circuit, a great error would be introduced by endeavouring to measure the average product by the split dynamometer method; but there is no such error here.

Hence Mr. Blakesley's method is correct, however great may be the magnetic leakage. It must be remembered, however, that I have neglected eddy currents in the copper and iron; and I assume magnetic permeability to be constant *during a cycle*. If I had time, I could show that in alternating-current calculations there are other very important uses of the fact that $\int_t^{\tau+t} x\dot{x}\,.\,dt=0$, if x is any periodic function of the time.

Now at full loads on this transformer it is perfectly obvious that currents, lags, and powers are immensely altered by this small amount of leakage which I have introduced as possible. The currents are ten times as great, and the lags are utterly different from what they have been supposed to be.

TABLE I.—No Magnetic Leakage, or M = 1·5.

ρ.	A.	A′.	ε.	ε′.	P.	P′.	Effic.	a′.
			°	°				
∞	0·1000	0	89·9	179·9394	·0850	0	0	100
99·9	0·1412	0·9991	44·97	179·9395	50·06	49·85	99·59	99·82
49·9	0·2232	1·9965	26·50	179·9395	99·95	99·45	99·49	99·61
9·9	0·9956	9·913	0·05	179·9401	497·8	486·4	97·70	99·14
4·9	1·963	19·61	0	179·9409	981·7	942·3	95·98	96·08
0·9	9·09	90·95	0	179·9468	4545	3722	81·70	81·85
0·4	16·67	166·6	0	179·9519	8335	5551	66·61	66·64
0·1	33·33	333·3	0	179·9610	16667	5554	33·33	33·33
0	50	500	0	179·9708	25000	0	0	0

TABLE II.—One and one-third per cent Magnetic Leakage, or M = 1·48.

ρ.	A.	A′.	ε.	ε′.	P.	P′.	Effic.	a′.
		°	°	°				
∞	·1000		89·9	179·939	·085	0	0	100
99·9	·1412	0·9858	46·4	181·5	48·61	48·54	99·70	98·49
49·9	·2228	1·968	29·5	182·98	96·96	96·65	99·68	95·98
9·9	·9627	9·458	20·4	194·8	451·2	442·8	98·13	93·63
4·9	1·741	17·15	30·4	204·85	750·8	720·7	96·01	84·05
0·9	3·370	33·28	68·2	247·6	628·4	498·2	79·27	29·95
0·4	3·658	36·11	77·6	257·33	392·8	260·7	66·37	14·44
0·1	3·726	36·78	83·7	263·55	204·4	53·75	26·29	3·678
0	3·739	36·91	85·8	265·7	136·9	0	0	0

Added May 23rd, 1891.

In the discussion of this paper it was obvious that I had not at sufficient length made known what I meant by magnetic "*leakage*." It was owing to this, no doubt, that my introduction of the idea of the importance of leakage was looked upon as academic merely. Again, my use of the symbols L, M, and L′ did not seem to be understood, nor what they had to do with a transformer. It is therefore necessary that I should say more fully, but not more definitely than in the paper, that as M is always less than $\sqrt{LL'}$ owing to magnetic leakage, I define leakage as the value of y where

$$M=(1-y)\sqrt{LL'}.$$

Hence, if $L=P^2\frac{4\pi}{10}\frac{a\mu}{\lambda}$ or P^2m, say, then $L'=S^2m$ and $M=PSm(1-y)$.

Again, the method of treatment to which I object is to state the equations as

$$V=RC+P\frac{dI}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$0=R'C'+S\frac{dI}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

I affirm that the induction I of equation (1) is a very different thing from the I of equation (2). As the old Maxwell method of writing the equations does not seem to be understood, I wish to make it clear that if I did use the induction I would use I_s in equation (1) and I_s in equation (2), where

$$I_p=PmC+Sm(1-y)C',$$
$$I_s=Pm(1-y)C+SmC'.$$

This is assuming that the number of ampere-turns which produces the effective induction through the primary is $PC+SC'(1-y)$; and the number of ampere-turns which produces the effective induction through the secondary is

$$PC(1-y)+SC'.$$

In fact, as I stated clearly when reading the paper, y is the fractional portion of the field due to primary current which escapes the secondary winding, or it is the fractional portion of the field due to the secondary current which escapes the primary winding. The reasoning is the same even if a small difference be supposed to exist between the y of the first and the y of the second equation; that is, if M is not the same in the two circuits. I submit that in the absence of any prior definition of "leakage" this is simple and reasonable. Dr. Sumpner's "leakage" is a very different thing. He said, let $I_p = I_s(1+x)$, then x is the magnetic leakage.

Of course, on any reasonable assumption of alterations in μ, the permeability of iron, or on the most reasonable assumption that μ really is constant during a quickly-performed cycle, Dr. Sumpner's x varies greatly during the cycle. I cannot give a physical meaning to x. My "leakage" y does not vary if μ is constant during a cycle, and I have given a perfectly definite physical meaning to it.

It may be well to add, here, why I think it reasonable to assume μ constant during a cycle.

1st. I have shown that if there is any leakage, the use of equations (1) and (2) is quite unreasonable.

2nd. If there is any leakage, it is impossible to treat the subject mathematically unless some assumption is made concerning μ.

3rd. It is quite certain that the law of variation of μ when the cycle is very slowly performed must be quite different from the law when the cycle is very quickly performed, as it always is in practice.

4th. The analogies between magnetic stress and strain and ordinary stress and strain in materials are well established. Now every material exhibits strain hysteresis when slowly loaded and unloaded, and exhibits no hysteresis whatever when the loading and unloading are very quickly performed. Even the most inelastic of materials will transmit a musical note unchanged. Hence for years I have taught my students to look upon magnetic hysteresis as very important when cycles are slowly performed and as unimportant when cycles are very quickly performed. Unless on this assumption, how is it that there is so little heating of the iron of a transformer by hysteresis even when transforming the largest amounts of energy; and such heating as there is must be partly due to eddy currents. I therefore maintain that μ constant during a cycle (and this means that my "leakage" is constant during a cycle) is the only reasonable assumption that can be made in the present state of our knowledge.

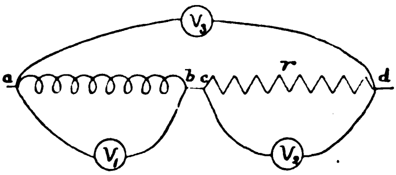
But even if this reasonable assumption of no hysteresis and of the constancy of μ during a cycle be denied me, and if I must assume the possibility of its being wrong, still $\frac{LL'-M^2}{L'}$ must be more nearly constant than μ; for it is equal to $2Ly$. And if μ increases and therefore L increases, y will certainly diminish, and if L diminishes y will certainly increase.

XXIV. *Alternate Current and Potential Difference Analogies in the Methods of Measuring Power. By* Prof. W. E. Ayrton, *F.R.S., and* W. E. Sumpner, *D.Sc.**

I.

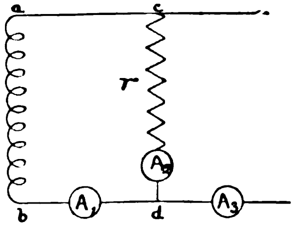
In a paper read by us before this Society on March 6th it was pointed out that for every problem involving alternate *P.Ds. in series* there was an analogous problem involving alternate *currents in parallel.* This general proposition tells us that we can transform each of the P.D. equations given, for example, in our paper on "The Measurement of the Power given by any Electric Current to any Circuit," read before the Royal Society, April 9th, 1891, into a current equation, and so transform our method of calculating power by the measurement of three P.Ds. into a method of calculating power by the measurement of three currents.

Fig. 1.



* Read June 12, 1891.

Fig. 2.



Such a transformation of our equations has been recently carried out by Dr. Fleming in the 'Electrician,' for May 8th, and the method he arrives at as well as the three-voltmeter method of which it is an analogue are seen in figures 2 and 1. If V_1, V_2, V_3 be the readings of the voltmeters in figure 1, and A_1, A_2, A_3 the readings of the three ammeters in figure 2, and r the resistance of the non-inductive portion of the circuit *cd* in each case, then the mean watts given to *ab* are respectively, whatever be the nature of the circuit *ab*, or of the current

$$\frac{1}{2r}(V_3^2-V_1^2-V_2^2)$$

and

$$\frac{r}{2}(A_3^2-A_1^2-A_2^2).$$

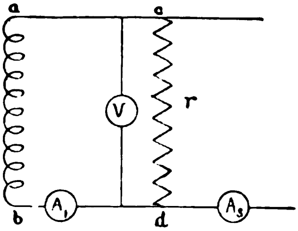
The three-ammeter method has the advantage over the three-voltmeter method, in that the dynamo need not give a larger P.D. than that necessary to send the current through *ab*; it is inferior to the three-voltmeter method in that while it is possible to measure V_1, V_2, and V_3 rapidly in succession by using only one voltmeter, it is, of course, impossible to use only one ammeter to measure A_1, A_2, and A_3 without constantly interrupting the circuit, and hence it would be necessary to accurately calibrate three instruments if this current method were employed.

But the main objection to this current method is that, as Dr. Fleming points out, it does not possess the accuracy of our three-voltmeter method of measuring power. For in order that this three-current method may give accurate results it is necessary to assume, to quote from our Royal Society paper, "the entire absence of self and mutual induction from a circuit some portion of which is necessarily of a solenoidal form."

It is possible, however, to obtain a current analogue of our three-voltmeter method which shall have the accuracy of the three-voltmeter method itself. And as the general proposition given in our previous paper tells us that the current analogues of P.D. arrangements *in series* are current arrangements *in parallel*, it follows that with this other method also the dynamo will not be required to produce a greater P.D. than that necessary to send the current through the circuit the power given to which we desire to measure.

The method is as follows:—In parallel with the circuit *ab* (fig. 3) the power given to which we wish to measure connect a non-inductive resistance of r ohms (in circuit with which no instrument is placed, which would necessarily make the so-called non-inductive branch more or less inductive). Let A_3, A_1, and V be the readings of the two ammeters and the voltmeter placed as shown, then, from the equations given in our Royal Society paper, it follows at oncethat the mean watts given to *ab* are

Fig. 3.



$$\frac{r}{2}\left\{A_3^2-A_1^2-\left(\frac{V}{r}\right)^2\right\}.$$

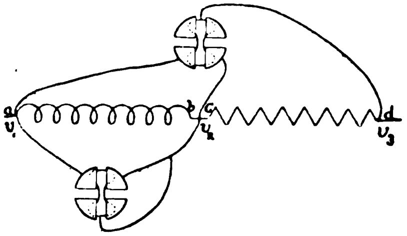
It is interesting to notice that if *ab* were the primary coil of a transformer, it would be when the load on the secondary was small, that is when the current passing through *ab* was small, that it would be most difficult, on account of lag, to measure with ordinary methods the power given to *ab*. But that is exactly the case when it is most easy to use our one-voltmeter and two-ammeter method, since when the dynamo has to send little current through *ab* there is little objection to requiring it to send a current through *cd* in parallel with *ab*.

If the voltmeter (fig. 3) be a hot wire instrument, then, since an appreciable current will pass through this voltmeter, r must be taken as the parallel resistance of *cd* and of the voltmeter. It is important to observe, however, that there is no necessity to know either of these resistances separately, since the value of r can be determined when the three instruments A_1, A_3, and V are relatively calibrated thus:—First, break the circuits of *cd* and of the voltmeter, and compare the deflexion of A_3 with A_1; this calibrates ammeter A_3 relatively to ammeter A_1, the calibration of which we will assume to be correct; second, close the circuits of *cd* and of the voltmeter, but break the transformer circuit *ab*, A_3 is now in series with the parallel circuit containing *cd* and the voltmeter. The value of *r* is therefore at once known, since *r* must equal the volts as read by V divided by the amperes as read by A_3.

II.

As an illustration of the general proposition to which we have referred, it may interest the Members to see what are the other analogies that we have traced out between alternate P.Ds. in series and alternate currents in parallel in connexion with the measurement of power.

Fig. 4.



Mr. Blakesley's method, communicated to this Society in February of this year *, is the current analogue of our electrometer method of 1881. For with the electrometer method (fig. 4) we make two measurements, one giving us the mean value of

$$(v_1-v_2)\left(v_3-\frac{v_1+v_2}{2}\right),$$

the other the mean value of

$$\frac{(v_1-v_2)^2}{2},$$

v_1, v_2, v_3 being the instantaneous values of the potentials.

---

* *Ante*, p. 106.

Then we take the difference, and so get the mean value of

$$(v_1 - v_2)(v_2 - v_3),$$

which is equal to r times the mean watts given to *ab* (fig. 4).

With Mr. Blakesley's method two measurements are made, one with a split dynamometer (fig. 5) giving the mean value of

$$\alpha_1 \alpha_3,$$

and the other with an ammeter giving the mean value of

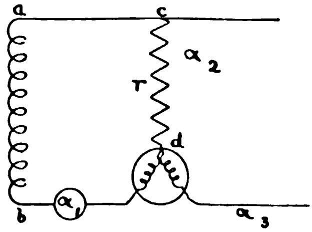
$$\alpha_1^2,$$

α_1, α_2, α_3 being the instantaneous values of the currents. Then the difference is taken, and so the mean value of

$$\alpha_1(\alpha_3 - \alpha_1) \text{ or } \alpha_1 \alpha_2$$

is obtained, and this is equal to $\frac{1}{r}$ times the mean watts given to *ab* (fig. 5).

Fig. 5.



It is important to notice that as no instrument is inserted in the non-inductive circuit *ab* (fig. 5), this method of Mr. Blakesley's has exactly the same accuracy as the electrometer method.

III.

The electrometer measurements may, as we pointed out some years ago, be varied; and by making the connexions as seen in fig. 6, we can obtain from the two readings the mean values of

$$(v_2 - v_3)\left(v_1 - \frac{v_2 + v_3}{2}\right)$$

and

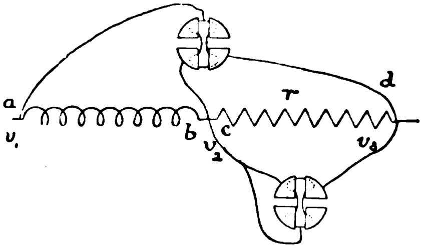
$$\frac{(v_2 - v_3)^2}{2};$$

then, taking the difference, we get the mean value of

$$(v_2 - v_3)(v_1 - v_2),$$

which, as before, is equal to r times the mean watts given to *ab*.

Fig. 6.



Mr. Blakesley has also pointed out, in his paper read in February of this year (*ante*, p. 106), that the current measurements may be varied and the apparatus arranged as seen in fig. 7. The two measurements now give respectively the mean values of

$$\alpha_2\alpha_3$$

and

$$\alpha_2^2;$$

therefore the difference gives the mean value of

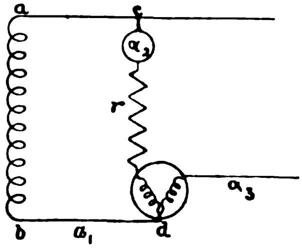
$$\alpha_2(\alpha_3-\alpha_2),$$

that is, the mean value of

$$\alpha_2\alpha_1,$$

which is equal to $\frac{1}{r}$ times the mean watts given to *ab*.

Fig. 7.



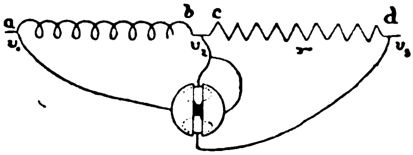
While, however, our second method of using the electrometer (fig. 6) gives the answer with the same accuracy as the first method (fig. 4), Mr. Blakesley's second method of joining up the dynamometers in figure 7 introduces self-induction into a circuit which ought to be entirely non-inductive, and so it does not give the answer with the same accuracy as his first method (fig. 5).

IV.

The modification of our electrometer method suggested to one of us by Mr. L. Atkinson, while he was a pupil at the

Finsbury Technical College, and afterwards carried out by MM. Blondlot and Curie, of making an electrometer with two needles for the measurement of power, is the exact analogue of the wattmeter method. For with the double-needle electrometer (fig. 8) we obtain from a single reading

Fig. 8.



the mean value of

$$(v_1 - v_2)(v_2 - v_3),$$

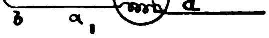
which is equal to *r* into the mean watts given to *ab*. Similarly with the wattmeter (fig. 9) we obtain from a single reading the mean value of

$$\alpha_1\alpha_3,$$

which is proportional to $\frac{1}{r}$ into the mean watts given to *ab*.

While, however, the double-needle electrometer gives us the answer with perfect accuracy, the wattmeter method is liable to inaccuracy from the circuit *cd* not being strictly non-inductive.

Fig. 9.



Some years ago, at a meeting of the Institution of Electrical Engineers, one of us published the formula for the error made in using a wattmeter to measure the power given by an alternate current to an inductive circuit. Not wishing to cumber the pages of scientific periodicals with elementary mathematics, it was thought sufficient merely to state this formula without publishing a proof. But as our formula has now been introduced into text-books, and as the appropriation thereof by the writer of a well-known treatise has led him to supply a proof of it involving an appalling array of mathematical equations, we venture to offer a proof which, although very simple, is perfectly accurate. We are the more induced to do this because we find that this formula, and its proof for the error due to self-induction in the supposed non-inductive portion of the circuit *cd*, apply equally well to all the nine methods of measuring power illustrated in figures 1 to 9 of this paper.

The formula employed with such methods for giving the mean watts, whether it involves the reading of one instrument, as in the case of the wattmeter (fig. 9), or of two instruments, as with the methods illustrated in figures 4, 5, 6, 7, and 8, or of three instruments, as with the methods illustrated in figures 1, 2, and 3, gives with perfect accuracy r times the mean product of two currents, or $\frac{1}{r}$ times the mean product of two P.D.s. Whether this mean product is directly proportional to the mean watts given to *ab* depends in all the nine cases on the following consideration :—

The mean product between two currents which are sine functions of the time is, as every student now knows, equal to half the product of their maximum values into the cosine of the phase angle between them. Therefore if the angle of lag between the current in *ab* and the P.D. between its terminals be θ, and the angle of lag between the current in *cd* and the P.D. between its terminals be ϕ, and if the maximum values of the currents in these two circuits be A_1' and A_2' respectively, and the maximum values of the P.D.s at the terminals of these circuits be V_1' and V_2', it follows that the formula used to measure the watts in the cases 2, 3, 5, 7, and 9 gives

$$r\frac{A_1'A_2' \cos(\theta-\phi)}{2},$$

and in the cases 1, 4, 6, and 8,

$$\frac{V_1'V_2' \cos(\theta-\phi)}{2r}.$$

But what we want to measure is the mean product of the current in *ab* into the P.D. between its terminals, and this product equals

$$\frac{A_1'V_1'\cos\theta}{2}.$$

But

$$rA_2' = V_2'\cos\phi;$$

and in the methods illustrated in the figures 2, 3, 5, 7, and 9

$$V_2' = V_1',$$

while in the methods illustrated in the figures 1, 4, 6, and 8

$$A_2' = A_1';$$

therefore in all the nine cases

$$\frac{\text{Apparent watts}}{\text{True watts}} = \frac{\cos(\theta-\phi)\,.\cos\phi}{\cos\theta}$$

$$= \frac{1+\tan\theta\,.\tan\phi}{1+\tan^2\phi}. \quad . \quad . \quad . \quad (1)$$

The circuit *cd* need only possess a self-induction *l*, even if an ammeter or dynamometer form part of it, but the circuit *ab* may have mutual induction and capacity as well as self-induction. If therefore we write expression (1) in the form

$$\frac{1+\frac{Lp}{R}.\frac{lp}{r}}{1+\left(\frac{lp}{r}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as we did in 1888, where p equals 2π times the frequency, it must be remembered that while l and r are the true values of the self-induction and resistance of *cd*, L and R are only the *effective* self-induction and resistance of *ab*. Hence, as Mr. Blathy suggested in the 'Electrician' for 1888, it is better to leave our expression for the ratio of the apparent to the true watts in the general form as given in (1) rather than to put it in the derived form as given in (2).

ϕ will generally be positive if the resistance of *cd* is small; but, if *cd* contains a doubly-wound high resistance-coil, as is generally the case when *cd* is the fine-wire circuit of a watt-meter, then it is quite possible to make ϕ positive, nought, or negative. θ may, of course, be also positive, nought, or negative, depending on whether the self and mutual induction effects preponderate or not over the capacity effect. It is therefore possible to have either θ or ϕ, or both, positive or negative.

The apparent watts will therefore be :—

too large if θ and ϕ be both of the same sign and $\theta > \phi$;

too small if $\left\{\begin{array}{l}\text{(1) } \theta \text{ and } \phi \text{ be both of the same sign and } \theta < \phi; \\ \text{(2) } \theta \text{ or } \phi \text{ be of different signs ;}\end{array}\right.$

correct if $\left\{\begin{array}{l}\text{(1) } \theta \text{ and } \phi \text{ be equal ;} \\ \text{(2) } \phi \text{ be nought.}\end{array}\right.$

Now ϕ can be made very small in one or other of three ways :—

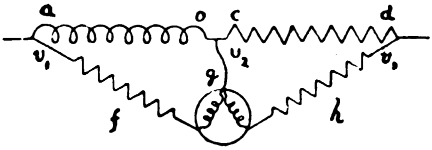
1. Use some method of testing, like that shown in figures 1, 3, 4, 5, 6, and 8, which does not require any measuring instrument to be placed in the non-inductive circuit *cd*.
2. Use a wattmeter (fig. 9), and make the capacity of the stationary doubly-wound resistance-coil exactly balance the self-induction of the suspended and the stationary coils.
3. Make the resistance of the fine-wire circuit, *cd*, of the wattmeter small. For with a given P.D. between the terminals of *cd* the same deflexion of the measuring instrument can be obtained for different values of the resistance of *cd* if we make the number of turns in the coil or coils of the measuring instrument in *cd* proportional to the resistance *cd*. But the self-induction of the coil or coils is proportional to the square of the number of turns, and therefore proportional to the square of r for a given deflexion of the measuring instrument. Hence tan ϕ can be made as small as we like for a given value of p by making the resistance of *cd* small.

This suggests a current method of measuring the power given to any circuit which is no more wasteful of power than the methods shown in figures 2, 3, 5, and 7 ; and which, although not so accurate as those shown in figures 3 and 5, is as accurate as those shown in figures 2 and 7. The method is simply to use a wattmeter (fig. 9), but having both its coils made of thick wire, or, as this may be called a split dynamometer, the method consists in using a split dynamometer having one of its coils in the circuit *ab* (the power given to which we desire to measure) and the other coil in a circuit *cd* parallel to *ab*. The power will be given at once by *r* times the reading of the instrument and with but a very small error if *r* be small.

V.

Mr. Rimington has suggested a method of measuring the mean value of the product $(v_1-v_2)(v_2-v_3)$ (fig. 9) by means of a dynamometer, each of whose coils is in circuit with a high resistance, joined up as shown in figure 10.

Fig. 10.

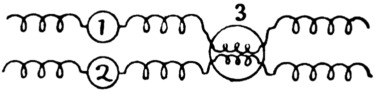


The objection to this method is as follows:—By making the time-constants of each of the circuits of the dynamometer *a f g*, *d h g* equal to one another, we can, no doubt, make the difference of phase in the two currents passing through the dynamometer exactly the same as the difference in phase between the current through *ab* and the P.D. at the terminals of *ab*; but we cannot make the currents through the dynamometer coils independent of the rate of alternation. Hence, if this instrument be employed for measuring the power given to *ab* in the way shown in figure 10, it must be calibrated for each rate of alternation of the current.

But although this defect exists in the employment of Mr. Rimington's high-resistance split dynamometer for the measurement of power, it can be used without error for measuring the phase-angle between *two P.D.s in series* by a method analogous to that employed by Mr. Blakesley for measuring the phase-difference between *two currents in parallel.*

Fig. 11 shows Mr. Blakesley's method: the dynamometers 1 and 2 give respectively the mean squares of the currents in the two circuits, and the dynamometer 3 the mean product of the two currents; and from the three readings we have, as is now well known,

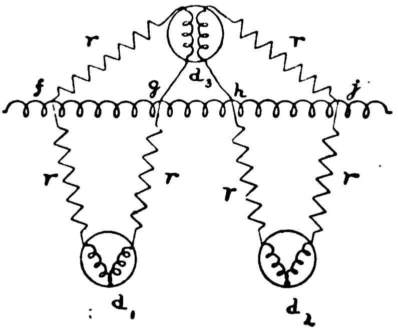
Fig. 11.



$$\cos\theta = \sqrt{\frac{\text{Square of reading of 3}}{\text{Reading of 1} \times \text{Reading of 2}}},$$

where θ is the angle of lag.

Fig. 12.



Now let fg and hj be two circuits in series (fig. 12), and let it be required to find the angle of phase-difference θ between the P.D. at the terminals of fg and the P.D. at the terminals of hj. Connect up the high-resistance dynamometer successively as shown, and let d_1, d_2, and d_3 be the three deflexions obtained.

Each circuit of the dynamometer consists of a coil of fine wire, and a non-inductive high resistance in series with it. Let r be the total resistance of each circuit, and let ϕ be the angle of lag between a current in either circuit of the dynamometer and the P.D. at its terminals.

Let V_1' and V_2' be the maximum values of the P.D. between f and g, and between h and j respectively, then

$$d_1 \propto \tfrac{1}{2}\left(\frac{V_1'}{2r}\right)^2 \cos^2\phi,$$

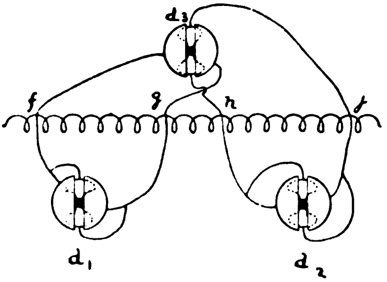
$$d_2 \propto \tfrac{1}{2}\left(\frac{V_2'}{2r}\right)^2 \cos^2\phi,$$

$$d_3 \propto \tfrac{1}{2}\left(\frac{V_1'}{r}\cos\phi \times \frac{V_2'}{r}\cos\phi\right)\cos\theta,$$

$$\therefore \quad \cos\theta = \sqrt{\frac{d_3^2}{16\,d_1 d_2}}.$$

Figure 13 shows the way in which the Blondlot and Curie double-needle electrometer can be successively connected up

Fig. 13.



so as to obtain, from the readings d_1, d_2, d_3, the angle of phase-difference between the P.D. at the terminals of $f\,g$, and the P.D. at the terminals of $h\,j$, the formula being, of course,

$$\cos\theta = \sqrt{\frac{d_3^2}{d_1 \times d_2}}.$$

If we desire to measure the angle of lag between the current in any circuit ab and the P.D. between its terminals, we can employ either the three-voltmeter method (fig. 1) or either of its analogues, viz. the three-ammeter method (fig. 2), or the one-voltmeter and two-ammeter method (fig. 3).

The formulæ giving the cosine of the lag angle for the three methods are, respectively,

$$\cos\theta = \frac{V_3^2 - V_1^2 - V_2^2}{2V_1V_2},$$

$$\cos\theta = \frac{A_3^2 - A_1^2 - A_2^2}{2A_1A_2},$$

$$\cos\theta = \frac{A_3^2 - A_1^2 - \left(\frac{V}{r}\right)^2}{2A_1\frac{V}{r}},$$

V_1, V_2, V_3, A_1, A_2, A_3, and V being the readings of the instruments in the different cases.

XXV. *The Expansion of Chlorine by Light as applied to the Measurement of the Intensity of Rays of High Refrangibility.* *By* Dr. A. Richardson, *Lecturer on Chemistry, University College, Bristol*[*].

[Plates III. & IV.]

It has been shown by Budde (Phil. Mag. iv. 1871; Pogg. *Ann.* Ergbd. vi. 1873) that when chlorine is exposed to the influence of sunlight, an expansion of the gas occurs which is independent of the direct heating-effects due to the light; the volume to which the gas first expands is maintained during exposure provided that the intensity of the light remains constant, contraction to the original volume taking place when the gas is shaded. He further found that the rays of high refrangibility were influential in promoting this change, no expansion being occasioned by the rays at the red end of the spectrum. The application of this property of chlorine to the measurement of the "actinic"* intensity of light was suggested by Budde many years ago, but no further steps appear to have been taken in this direction.

---

[*] Read June 26, 1891.

Some experiments on which I am at present engaged have rendered it necessary that the actinic intensity of light should be measured during periods of many months together, and it seemed possible that the expansion of chlorine by light might be applied to this purpose. As, however, the researches of Bunsen and Roscoe (Trans. Roy. Soc. 1887, p. 381) led them to the conclusion that no change in volume occurred in chlorine, when exposed to light, other than that due to direct heating-effects, it became necessary to repeat some of Budde's experiments so as if possible to decide this point. In order to do this a differential apparatus was taken, consisting of two bulbs of 160 cubic centim. capacity each and connected by a capillary tube containing strong sulphuric acid; any change in pressure of the gas in either bulb was indicated by the movement of an index of air contained in the capillary.

The bulbs were first filled with dry air and placed in a glass tank through which a current of cold water circulated; under these conditions it was found that no movement of the index took place on exposure to bright sunlight. The air in one of the bulbs was then replaced by dry chlorine, and both bulbs were again exposed under precisely similar conditions; an immediate expansion of the chlorine took place, causing the index to move through a distance of from 20 to 30 centim., but on shading the bulbs the index returned to zero. On interposing variously coloured glasses between the bulbs and the source of light, it was found that with cobalt glass the index receded 15 to 20 centim. from the zero-point, whilst even faintly yellow glass as well as ruby glass caused it to return to zero. It was further found that the radiations from a cannon-ball at a temperature just below a red heat produced practically no movement of the index when the bulbs were exposed in the air at a distance of from 1 to ·5 metre.

Budde's observations being thus confirmed, it was next necessary to determine how far the expansion of chlorine is proportional to the actinic intensity of the light to which it is exposed. In order to do this a series of measurements of the expansion of the gas were made, whilst at the same time the intensity of the light to which the gas was exposed was determined by means of an actinometer.

---

* The term "actinic" is used for brevity to denote rays at the violet end of the spectrum.

The expansion of the gas was measured in a differential arrangement, consisting of two tubes of 55 cubic centim. capacity and 10 centim. in length; these were connected with a horizontal gauge graduated in ·5 centim., provided with a small bulb at each end. These bulbs and also the gauge contained strong sulphuric acid, a short column of air being introduced to serve as index.

The tubes to be exposed were suspended in a box, which could be placed at any required angle so as to face the sun, and when filled with dry air were found to be equally heated; the bulbs containing the acid were in all cases protected from the light. One tube was then filled with dry chlorine and the acid up to the index was saturated with the gas.

The chlorine used in this and subsequent experiments was prepared by the action of hydrochloric acid on potassium bichromate; traces of hydrochloric acid were removed by passing the gas through U-tubes containing solutions of chromic acid, and subsequently dried by means of sulphuric acid. In order to determine the actinic intensity of the light, a modified form of Bunsen and Roscoe's pendulum-actinometer was used (Trans. Roy. Soc. 1862, p. 139); in this method paper coated with silver chloride is exposed to light, and the time required to produce a degree of darkening equal to a standard tint is measured. The intensity of light necessary to produce this tint in one second of time is taken as the unit of intensity. The measurements were made in the open air, care being taken that the air- and chlorine-tubes were exposed to light in the same plane as the sensitive paper. When the maximum expansion of the chlorine was reached at any one time, as shown by the index remaining stationary, the intensity of the light was measured by exposing the sensitive paper. Two series of observations were made under widely different conditions of light; one on June 8, which was a cloudy day, the other on June 9, on which the sun was shining. The results of these measurements are given in the following table:—

First Series.

Time.	I. Intensity of light (unit =1).	II. Expansion of Cl, expressed in centimetres on scale.	III. Ratio $\frac{I}{II}$.	IV. Recalculation of intensities from observed expansion.
m. s.				
2 50	0·0242	1·95	0·0124	0·0263
2 55	0·0571	3·20	0·0178	0·0432
3 0	0·0461	3·20	0·0144	0·0432
3 5	0·0291	1·87	0·0155	0·0252
3 15	0·0196	1·40	0·0140	0·0188
4 0	0·0382	2·70	0·0141	0·0364
Second Series.				
10 45	0·3758	28·15	0·0133	0·3799
10 50	0·1022	7·95	0·0128	0·1073
11 0	0·1693	12·2	0·0138	0·1647
11 5	0·3240	27·95	0·0116	0·3773
11 15	0·4381	26·20 ?	0·0167	0·3537
11 30	0·3348	28·0	0·0119	0·3779
11 35	0·3212	29·95	0·0107	0·4042
11 36	0·4457	30·80	0·0144	0·4158
12 20	0·0779	6·15	0·0126	0·0890
12 30	0·1210	7·70	0·0157	0·1040
12 32	0·1087	7·65	0·0142	0·1032
12 35	0·1133	9·25	0·0122	0·1249
4 0	0·1108	10·75	0·0103	0·1451
4 5	0·1141	10·05	0·0113	0·1357
4 15	0·1150	7·75	0·0148	0·1046

In column I. the intensity of the light is given as measured by the actinometer. In column II. the expansion of the chlorine, as shown by the movement of the index from zero, is given in centimetres; the zero-point, which was determined from time to time by shading the tubes, was found to remain nearly stationary throughout. The calculated intensity, in terms of the movement of the index through 1 centim., is given in column III., and from the mean value so obtained the intensity is recalculated in terms of the observed expansion (column IV.). In making these observations great difficulty was experienced owing to the constant variation in the intensity of the light; and as the maximum movement of the index was only reached after some time, these sudden variations, which were registered accurately by the actinometer, did not produce a corresponding change in the position of the index. The close agreement between the results obtained directly by the actinometer and those calculated from the expansion of the gas lead, however, to the conclusion that the change in volume in chlorine gas is directly proportional to the actinic intensity of the light to which it is exposed, although the friction of the acid in the gauge tends to smooth out the more sudden variations.

By the aid of such a differential apparatus as that above described, to which the name of "chlorine actinometer" may be applied, the intensity of the light can be read directly from the gauge, and it is thus found possible to study other light-effects under constant conditions of intensity by making exposures whenever the index stands at a given point.

It was next necessary to inquire how far dilution with air influenced the amount of expansion of chlorine when exposed to light. To do this a differential apparatus, shown in fig. 1, Plate III., was used, in which the gas to be exposed was contained in the tubes A and B, 15 centim. in length and about 3 centim. in diameter. Two glass stop-cocks, C and D, were sealed on to A, the lower one, D, being connected with a graduated pipette E containing air and dipping under strong sulphuric acid contained in F. The graduated capillary tube G was filled with strong sulphuric acid, as also the bulbs H and I, the movement of a short column of air serving to indicate any change in volume in either tube. The tube A was first filled with chlorine, whilst B contained dry air throughout the experiment. Both tubes were then exposed to light and the expansion of the chlorine determined, the intensity of the light being at the same time noted by means of the chlorine actinometer; the tubes were then shaded and the zero-point found. The stop-cocks C and D were then opened, and by blowing air in at the side tube J a measured volume of acid was caused to rise in E, thus forcing an equal volume of air into A. Chlorine being expelled at C; C and D were then closed, and, after the gas had thoroughly mixed, the tubes were again exposed to light of the same intensity as in the first case: this process was repeated until all the chlorine was replaced by air. Knowing the capacity of A and the proportion of air introduced at each operation, the amount of chlorine could be calculated. Two series of experiments were made under different conditions of light, the intensity for each series remaining nearly constant. The results are given in the next table.

First Series.

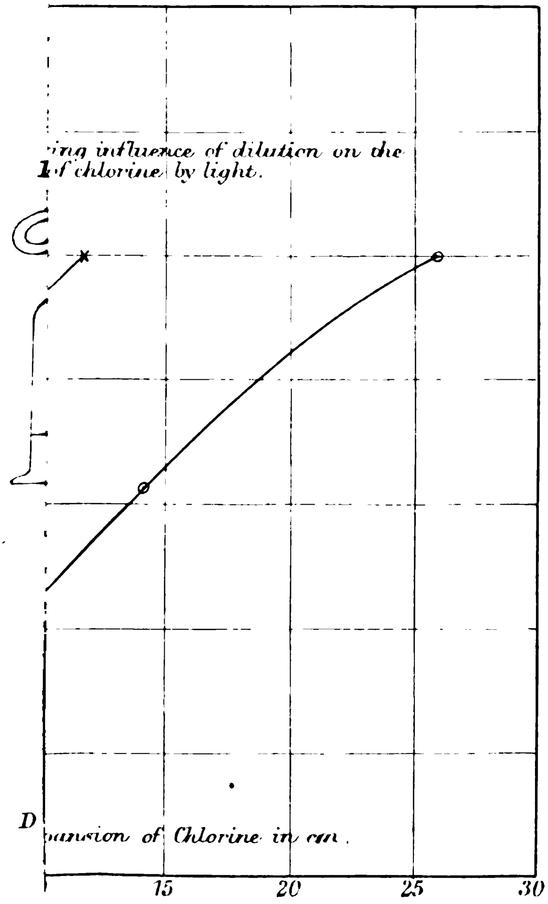
No.	Expansion of Cl in centim.	Per cent. of Cl in gas.	Expansion of Cl in actinometer.	Corresponding intensity of light (unit = 1).
1.	26·5	100	8·0	·1107
2.	14·5	63·7	7·9	·1092
3.	9·25	41·0	17·9	·1092
4.	5·25	26·1	17·9	·1092
5.	4·75	7·1	7·9	·1092
6.	2·0	2·0	7·9	·1092
7.	0·5	0·5	7·95	·1100
8.	0·0	0·1	7·9	·1092

Second Series.

No.	Expansion of Cl in centim.	Per cent. of Cl in gas.	Expansion of Cl in actinometer.	Corresponding intensity of light (unit = 1).
1.	11·7	100	3·55	·0491
2.	6·25	63·7	3·5	·0484
3.	5·95	41·0	3·5	·0484
4.	4·35	11·3	3·3	·0456
5.	1·0	3·0	2·2	·0304
6.	0·15	0·8	3·4	·0466

It is seen from the double nature of the curve, fig. 5, Plate III., that the first effect of dilution is very marked, diminishing after a time, and finally increasing when only a small per cent. of chlorine remains in the tube. It was found, however, that chlorine was gradually given off (from the sulphuric acid which was saturated with the gas) to such an extent that, after complete expulsion of the chlorine by air, on again exposing the tube to light, after two days, an expansion of 5 centim. occurred. This is a point of some importance where the expansion of chlorine as against air is made a measure of the actinic intensity of light, as the

Fig. 5.


ing influence of dilution on the
of chlorine by light.
D
pansion of Chlorine in cm.
15
20
25
30

diffusion of chlorine from the sulphuric acid into the air-bulb would introduce a gradually increasing source of error*.

As it seemed possible that the sensitiveness of chlorine to light might be influenced by the temperature of the gas during exposure, an experiment was made in which the two tubes A and B, fig. 2, Plate III., were filled with chlorine, these were connected together by a tube which could be closed by means of a stop-cock F ; changes in volume being measured as before by means of the gauge C containing sulphuric acid and bubbles of air. The tube A was kept at a temperature of 14° C. by a current of water circulating through the outer tube D, whilst B was heated in the jacketing tube G G, by the vapour from chlorobenzene boiling in E at a temperature of 132° C. The pressure in the two tubes was at first equalized by opening the stop-cock F ; and when the temperature was constant in the two tubes (as shown by the index remaining stationary when F was closed), the tubes were alternately exposed to and shaded from the light. The following is the mean of a series of experiments made : —

Both tubes shaded, zero = 0.
Heated tubes exposed = 6·0 centim. expansion.
Both tubes shaded, zero = 0.
Cooled tubes exposed = 6·7 centim. expansion.
Both tubes shaded, zero = 0.

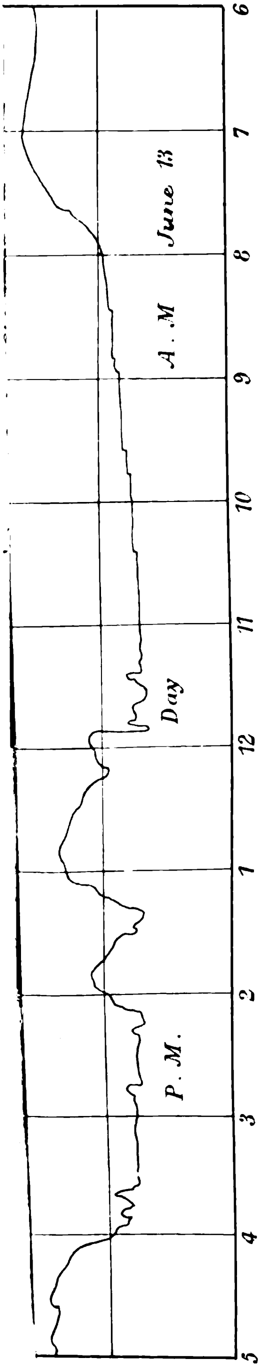
In the heated tube the illumination of the gas was interfered with by the partial refraction of the light, due to the condensation of the chlorobenzene on the sides of the jacketing tube, and further it is possible that the vapour of chlorobenzene may absorb a portion of the actinic rays. This probably accounted for the slight difference observed in the expansion of the gas in the two tubes. From this experiment it is concluded that the expansion of chlorine by light is practically unaffected within a range of temperature between 14^{c} and 138° C.

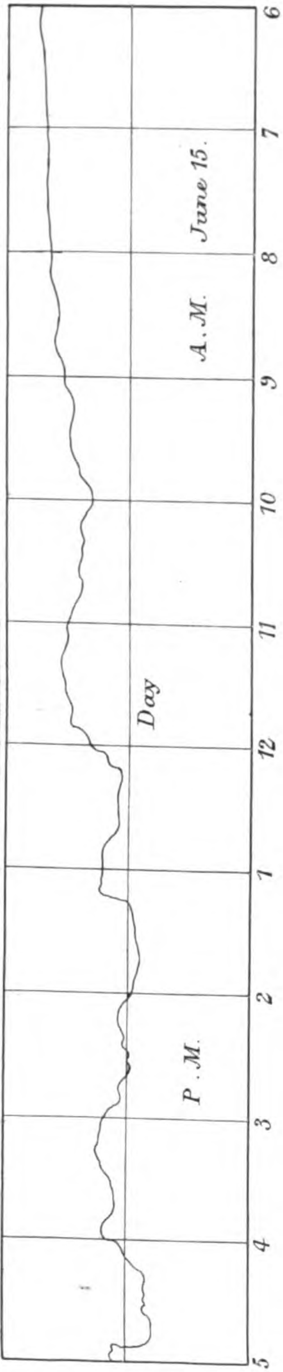
Having made these preliminary experiments, an apparatus was next devised whereby the automatic registration of the actinic intensity of light was effected by the expansion of chlorine. This was done by suspending a differential apparatus on the beam of a balance in such a manner that the flow of acid from one arm to the other produced a movement of the beam, which was communicated by means of a lever to a pen and was recorded on a rotating drum.

---

* In the chlorine actinometer it is found desirable to cut off communication between the two bulbs, when not in use, by means of a stop-cock.

As has already been stated, a differential actinometer, containing air in one bulb and chlorine in the other, gradually becomes inaccurate owing to the diffusion of chlorine into the air-bulb. This difficulty is avoided in the apparatus shown in fig. 3, Plate III., which consists of two differential arrangements A B C D and E F G H, suspended on the beam of the balance I; one is completely filled with dry chlorine, the tubes B and D being half filled with strong sulphuric acid saturated with the gas. The second contains dry air with sulphuric acid in F and H. One of the chlorine-bulbs A and one of the air-bulbs E are exposed to light, whilst the other chlorine-bulb C and air-bulb G are protected from the light by a covering of tin-foil, and hang in the box J, in which the two bulbs can swing freely. It will be seen that when A and E are exposed to the heating-effects of sunlight, the expansion of the gas in A causes acid to flow from B to D, whilst a corresponding expansion in E causes an equal weight of acid to flow from H to F. But the chlorine in A undergoes a further expansion, due to the actinic rays, causing an additional weight of acid to pass from B to D. A movement of the beam is thus produced which is communicated to the lever M, and is registered by means of a pen on a strip of curve-paper rotating on the drum N. The capacity of the bulbs A and E in this case is 273 cub. centim., that of G and C 319 cub. centim.; the tubes B and F and D and H are 15 centim. long and 1·25 centim. in diameter, the distance between the two sets of tubes being 35 centim. It is found desirable that the movement of the bulbs and beam should be as small as possible, and that the effect should be magnified by increasing the length of the lever. After the apparatus is blown together, care must be taken to remove all moisture by drawing dry air through the bulbs and tubes. The apparatus is exposed out of doors in a wooden box G, fig. 4, Plate III., the bulbs A and B passing through a hole in the





Mintern Bros. lith.

lid; these are protected by a glass shade, C, 45 centim. high and 20 centim. in diameter; this is mounted in a block of wood, D, so that when the box is in its normal position, the axis of the cylindrical shade points to the pole star. The two bulbs occupy a position equidistant from the sides of the shade and about midway between the top and bottom. The tube E serves as a ventilator and should be at least 2·5 centim. in diameter; the cap F protects the opening from rain, &c. The bulbs were tested, when filled with dry air, by exposing them to intense sunlight: it was found that, although the temperature within the shade varied from 15° to 40^{c} C., the compensation was so complete that practically no deviation from the straight line, described by the pen on the drum, was observed; the bulbs were fixed to the beam (as nearly as possible over the point of support) by means of cylinders of thin sheet brass, which were slipped over the acid tubes and screwed to the beam. The instrument was calibrated by means of the silver-chloride actinometer. The result of six series of observations gave the following values in intensity units corresponding to one division on the curve-paper :—

1 series mean of 11 observations ·021
2 ,, ,, 14 ,, ·029
3 ,, ,, 10 ,, ·043
4 ,, ,, 15 ,, ·025
5 ,, ,, 9 ,, ·033
6 ,, ,, 10 ,, ·029

Mean = ·030

In the following table the observation from one of the series is given :—

Intensity (unit = 1).	Value of 1 division in light units.	Recalculated intensity from curve.	Time of day in hours.
0·3758	0·0385	0·2925	10·75
0·1022	0·0328	0·1275	10·8
0·1693	0·027	0·1875	11·0
0·3240	0·035	0·2475	11·08
0·3348	0·029	0·3375	11·5
0·3210	0·030	0·3150	11·6
0·4457	0·039	0·3375	11·65
0·0779	0·017	0·1350	12·3
0·1087	0·021	0·1500	12·6

The curve from which the data are obtained is shown in fig. 1, Plate IV., the value of one division being equal to $\frac{1}{10}$ of a square therein represented.

In fig. 2, Plate IV., the results obtained above are multiplied by 100 and represented graphically. Fig. 3 represents the curves registered by the recording apparatus under varying conditions of light from June 13 to 16. It will be seen that such an instrument as this will record continuously the actinic intensity of the light under all conditions of weather throughout the year, and requires no attention further than winding the clock whereby the motion of the drum is maintained.

XXVI. *The Influence of Surface-Loading on the Flexure of Beams.* *By* Prof. C. A. Carus Wilson[*].

[Plate V.]

The practical treatment of the problem of beam-flexure at the present time is based on the hypothesis enunciated by Bernoulli and Euler[†], that the bending-moment is proportional to the curvature.

This assumes that the cross sections remain plane after flexure and neglects the surface-loading effect.

Saint-Venant has shown[‡] that the first assumption is untenable; but that, neglecting the surface-loading, Bernoulli's results are strictly true for one particular case of loading, that, namely, of a beam doubly supported and carrying a single isolated load, where, although the cross sections are distorted, the central displacement is zero.

I propose in this paper to describe some experiments made with a view to determining the actual state of strain in a beam doubly supported and centrally loaded, the surface-loading effect being taken into account.

The method of investigation adopted is based upon the following assumptions:—

---

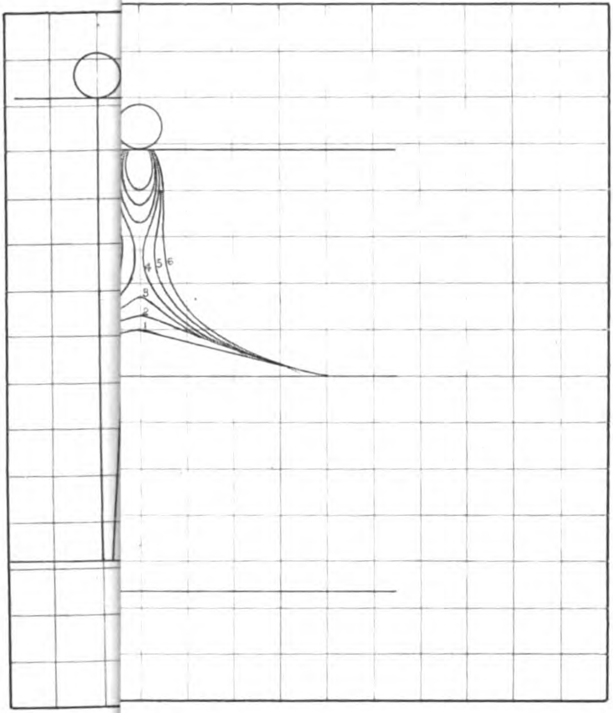
[*] Read June 26, 1891.

[†] Todhunter and Pearson's 'History of Elasticity,' vol. i.

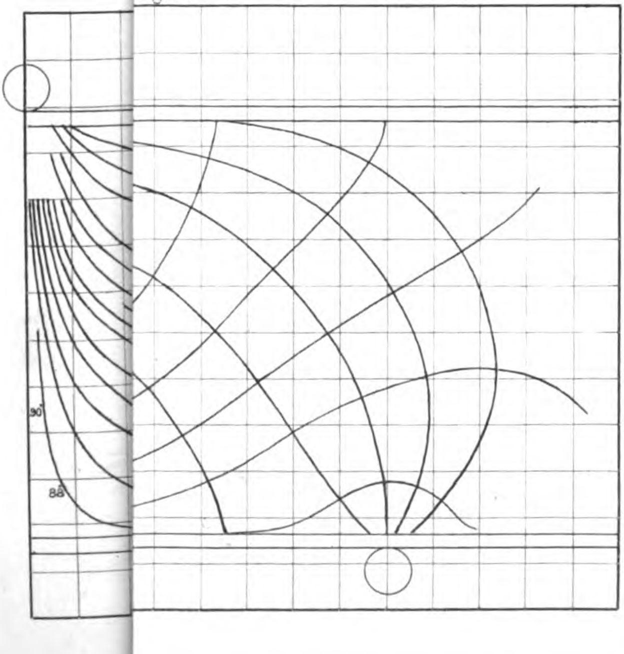
[‡] Pearson's 'Elastical Researches of Saint-Venant.'

Proc. Phys. Soc. Vol. 11. Pl. V.

ig. 2.



ig. 4.



(1) The true state of strain at the centre of the beam may be found by superposing on the state of strain due to bending only, that due to surface-loading without bending.

(2) The state of strain due to surface-loading only may be found, with close approximation to truth, by resting the beam on a flat plane instead of on two supports.

(3) The strains due to bending only may be obtained from the Bernoulli-Saint-Venant results ; viz.:—

(α) The stretch for any cross section varies as the distance from the neutral axis.

(β) The central axis is unstretched.

(γ) For the same point in different cross sections the stretch varies as the bending-moment.

Saint-Venant has dealt with the shearing-strains at some little distance from the load in the case of a beam doubly supported and centrally loaded* ; and Professor Pearson has shown† that, in the case of beams continuously loaded, the results of the Bernoulli-Eulerian theory can only be considered as giving approximate formulæ when the span of the beam is not less than ten times its depth‡.

The mathematical determination of the state of strain produced by the loading of a beam as it rests on a flat plane is one of considerable analytical difficulty.

MM. Lamé and Clapeyron have attempted the solution of a more general problem in their "Mémoire sur l'équilibre intérieur des corps solides homogènes."§ The object of this paper is stated to be "to investigate the way in which the interior of a body is affected by the transmission through it of the action of forces." Here they treat the problem of a solid extending to infinity on one side of a plane, on which is a given distribution of tractive load, and also of a solid contained between two parallel infinite planes. They obtain as a result a set of definite integrals giving the displacements, introducing a function involving the distribution of tractive load, from which the stresses may be deduced, but concerning which they add: "Les formules précédentes, pour être obtenues en séries numériques et immédiatement applicables, exigent la connaissance des valeurs d'un genre particulier d'intégrales définies, dont il ne nous paraît que les géomètres se soient encore occupés."

---

* Pearson's 'Elastical Researches of Saint-Venant,' §§ 69-99.

† Pearson, "On the Flexure of Heavy Beams subject to continuous systems of Load," Quarterly Journal of Mathematics, No. 93 (1889).

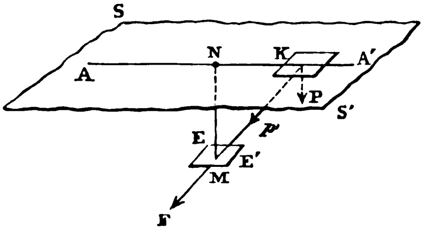
‡ Rankine assumed that the surface-loading effect might be neglected. See his 'Applied Mechanics,' § 311.

See also *Résumé des Leçons* &c. by Navier (Paris: Dunod, 1864), vol. i. p. 41:—"Observation sur le mode d'application et de distribution des forces qui font fléchir," where the same assumption is made.

§ Crelle's *Journal*, vol. vii. p. 145 *et seq.*

The most successful attempt at a solution of this problem is to be found in a more recent work by Professor Boussinesq, published in 1885*. The following is a brief account of the results obtained.

Fig. 1.



S S' being the surface of the solid (infinite below in length, width, and depth), M a point within, situate at a distance M N $=x$ below the surface, K any element of the surface, situate at the distance K M $=r$ from the point M, and subject to a given exterior pressure K P $=p$, having the component K P' $=p'$ along K M, the pressure which a plane element E E' taken through M parallel to the surface S S' will support, per unit of area, in consequence of the pressure p, will be found directed along the direction of K M produced, and will be equal to

$$\mathrm{MF}=\frac{3p'x}{2\pi r^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

* *Application des Potentiels à l'étude de l'Equilibre et du Mouvement des Solides élastiques* (Gauthier-Villars, Paris, 1885).

See also *Théorie de l'Elasticité des Corps solides*, Clebsch; translated and annotated by MM. de Saint-Venant and Flamant. (Paris: Dunod, 1883, p. 374, note to art. 46.)

If, as a particular case, the pressure KP $=p$ be normal, then $p'=p\cos \text{NMK}=p\,\frac{x}{r}$, and

$$\text{MF}=\frac{3px^2}{2\pi r^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, further, it is required to find the vertical component of MF, we have $(\text{MF})\,\frac{x}{r}$, or

$$\frac{3px^3}{2\pi r^5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The treatment of this particular problem is not pursued any further in this work; but Professor Boussinesq has kindly furnished me with a solution more nearly applicable to the case in point, and one which will be found to agree closely with the experimental results I had previously arrived at, and which are given later on.

Suppose there to be a uniform pressure p exerted over every element du of bearing-surface between two extremities A, A′ (see figure), having abscissæ $u=-\text{NA}=-a$, $u=\text{NA}'=+a$, and let $p=\text{P}du$, calling P the constant exterior pressure per unit of length $\text{AA}'=2a$.

The total pressure over unit of surface of an element E E′ will be, from equation (3),

$$\Sigma\frac{3px^3}{2\pi r^5}=\int\frac{3\text{P}x^3du}{2\pi r^5}=\frac{3\text{P}x^3}{2\pi}\int_{-a}^{+a}\frac{du}{r^5}$$

$$=\frac{3\text{P}x^3}{\pi}\int_0^a\frac{du}{(x^2+u^2)^{\frac{5}{2}}}\ [\text{NK}=u].$$

Putting $\frac{u}{x}=\alpha$, $du=x\,d\alpha$, we get as the normal pressure per unit of area on E E′,

$$\frac{3\text{P}}{\pi x}\int_0^{\frac{a}{x}}(1+\alpha^2)^{-\frac{5}{2}}d\alpha\ ;$$

or, very nearly, if x is much smaller than a,

$$\frac{3P}{\pi x}\int_0^\infty (1+\alpha^2)^{-\frac{5}{2}}\,d\alpha.$$

The value of the integral is $\dfrac{2\alpha^3+3\alpha}{3(1+\alpha^2)^{\frac{3}{2}}}$ or $\dfrac{2+\dfrac{3}{\alpha^2}}{3\left(1+\dfrac{1}{\alpha^2}\right)^{\frac{3}{2}}}$, which, between the limits $\alpha=0$ and $\alpha=\infty$, becomes $\frac{2}{3}$. Thus the pressure per unit of area on an element E E′ becomes $\dfrac{2P}{\pi x}$, or

$$0\cdot64\,\frac{P}{x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This expression has the form of that given below, though, inasmuch as the problem is not altogether the same as that treated experimentally[*], a difference in the coefficients is only what might have been expected.

The value of the integral between the limits $\alpha=0$ and $\alpha=\infty$ is, as has been stated, $\frac{2}{3}$, or 0·667. For $\alpha=5$, *i. e.* for $u=5x$ as the upper limit, the integral = 0·666, and for $u=2x$ the integral = 0·656; so that this solution is approximately correct for elements lying at a distance of ¼ the width of the beam from the point of contact.

Hence for a beam where the length A A′ is 5·5 millim., this solution would be applicable up to points lying at a distance of about 1·4 millim. from the top surface.

I have investigated the law up to within 0·5 millim. of the top surface, and find it to be

$$y=0\cdot726\,\frac{P}{x}.$$

The investigation of the state of strain in glass beams by means of polarized light was first suggested by Sir David Brewster[†], and his experiments are usually quoted as proving the truth of the Bernoulli-Eulerian theory of flexure. It is, however, easy to show experimentally that these experiments must have been made under conditions where the surface-loading effect was inappreciable; though very accurate reasoning on this point is impossible, as the drawings accompanying Sir David Brewster's paper are not to scale, and the span of the beams and the precise method of application of the loads are not indicated.

---

[*] The mathematical solution assumes the length of bearing A A′ on an infinite surface.

[†] Phil. Trans. 1818, p. 156.

M. Neumann developed a theory of the action of strained glass in the polariscope*, and found that the velocity of light in a medium is increased by compressing it. He bases his calculations on the measurement of the deflexions of glass beams supposed to obey the Bernoulli-Eulerian theory; the beams are doubly supported and centrally loaded, having the proportions 66 × 8·5 × 2, the latter being the depth. It is not in all cases stated what spans were employed, so it is impossible to say how far the results were influenced by surface-loading.

Professor Clerk-Maxwell † has examined the state of strain in pieces of unannealed glass of various shapes, the lines of equal intensity of strain being deduced from the isochromatic lines.

The lines of Principal Stress are found from those of Equal Inclination in the manner described later on in this paper.

It has already been pointed out ‡ that "Neither Neumann nor Maxwell seems to have remarked that the difference of the velocities of the ordinary and extraordinary rays depends solely on the maximum slide of planes perpendicular to the wave-front."

An important work on this subject is found in a paper by Dr. John Kerr §. He establishes the fact that "If a plate of glass, compressed or extended in one direction parallel to its faces, be traversed normally by two pencils of light, which are polarized in planes respectively parallel and perpendicular to the direction of strain, then both pencils are retarded by the strain in the case of compression, and both are accelerated by the strain in the case of tension." Also that "strain-generated retardations, absolute as well as relative, are sensibly proportional to the strain," thus confirming Wertheim's results.

---

* *Abhandlungen der k. Akademie der Wissenschaften zu Berlin*, 1841, vol. ii. pp. 50–61.

† Trans. Roy. Soc. Edinburgh, vol. xx. (1853) p. 117.

‡ Hist. of Elasticity, vol. i. p. 643.

§ Phil. Mag. October 1888.

Dr. Kerr employs in his experiments a bent glass beam, doubly supported and centrally loaded, having the ratio of span to depth* of 8·4 to 1, and assumed to obey the Bernoulli-Eulerian theory.

I would draw attention to the disagreement between the results arrived at by M. Neumann and Dr. Kerr, the former stating that the velocity of light in a medium is increased by compressing it, while the latter states that the velocity is diminished.

Dr. Kerr examined a beam having a span equal to 8·4 depths, and at a point where the surface-loading effect would be least; whereas M. Neumann examined a beam—span to depth ratio not stated—immediately under the load.

I can only attempt to account for the discrepancy by pointing out that if the span is diminished to less than four depths, the elements of glass that M. Neumann assumed to be in a state of squeeze are actually, as will be shown later, in a state of stretch.

The instrument with which the following experiments were made consists of a steel straining-frame in which the beam to be examined is placed; the beam rests—for flexure—on two steel rollers, and is loaded by a micrometer-screw which bears on a third central roller. The base of the frame is divided, from the centre, in divisions of 2 millim. so that the supports can be set for any required span. A micrometer-screw is placed in the base of the frame opposite the load, so that deflexions can be measured to one ten thousandth of an inch. Two screws in the two sides of the frame enable lateral pressure to be applied. The whole frame can be moved in any direction in its own plane, so that all parts of the beam may be examined. The optical arrangements consist of two nicols, of which the upper is provided with a graduated disk on which the angle of rotation can be observed; a microscope with micrometer-eyepiece can be fitted when it is desired to measure the fringes; circularly polarized light can be used when required.

---

* According to the figure.

The beams used were marked on one side with 2 millim. squares; they were covered with paraffin and marked in a dividing-engine and then etched; the lines thus formed enabled the position of dark bands to be determined with accuracy.

Proposition I.

If a beam of glass be laid on a flat surface and loaded across its upper surface, the shear at any point on the normal at the point of contact of the load is inversely proportional to the distance from the point of contact.

Experiment 1. A beam of annealed glass 61 millim. × 6·5 millim. × 20 millim. deep was placed in the steel straining-frame with its narrow side resting on a piece of thin paper.

A steel roller 2 millim. in diameter, 10 millim. long, was placed across the middle of the top surface and loaded by the screw.

The nicols were crossed and at 45° to the axis of the beam.

A quarter-wave mica plate was placed between the beam and the analyser, with the plane containing the optic axes at right angles to the length of the beam.

At that point *a* on the normal where the difference of phase between the ordinary and extraordinary pencils traversing the beam is equal and opposite to the difference of phase produced by the mica plate—the effect will be as if there were no strained glass between the two nicols, and there will therefore be a black spot as the nicols are crossed.

The position of this spot on the normal is plotted on a sheet of squared paper, and an ordinate parallel to the axis chosen to represent the shear.

A second quarter-wave plate is now superposed on the first, and the black spot consequently moves up the normal to where the shear is twice what it was at *a*; this point, *b*, is noted, the second mica plate removed, and the load reduced until the black spot with one mica plate is brought to *b*. In this way a series of points *a*, *b*, *c*, *d* on the normal are found at any one of which the shear is twice what it is at the point below.

Now it is proved later on that the strain at any point varies as the load on the beam; hence by taking the ordinate at *b* twice that at *a*, at *c* four times, and at *d* eight times, and so on, we get points on the curve of loading along the normal for the load that give a difference of phase at *a* equal to that of one-quarter wave-plate.

The results are plotted on Plate V. fig. 1: the observed points are indicated by circles, through one of which an hyperbola has been drawn taking the normal and the upper surface of the beam as asymptotes.

It will be seen that the six upper circles lie very nearly on the hyperbola.

It is clear that the upper surface of the beam is an asymptote only when the surface of contact between the beam and the roller is a line—making the stress there infinite; but in practice this cannot be so, the smallest pressure giving a bearing surface—as the roller indents the beam—making the stress there finite, *i. e.* the asymptote will be at some finite distance θ, say, above the point of contact, and θ will vary with the load. I have calculated below that with a load of 115·3 lb. on this same beam, the value of θ is 0·044 millim.

The apparently irregular position of the two lower points observed indicates the amount of error made in the assumption (2) above that the surface-loading effect may be found by substituting a flat plane instead of two supports.

This assumption would be correct only if the beam were of infinite depth and the surface-loading effect of the support infinitely small; here, however, the steel frame itself produces a surface effect, and this, added to that due to the load, makes the points observed lie off the hyperbola, which would be the true curve (as drawn) if the beam were of infinite depth.

The effect of the steel frame must be very small compared with that due to the load for points in the upper half of the beam. In drawing the hyperbola I have considered it as negligible at the centre of the beam; in other words, I consider that the correction of the position of the six upper points, required to allow for the surface effect of the frame, would not make them deviate seriously from the hyperbola.

It must be noted, however, that when the beam is resting on two supports the surface effect of the frame disappears, since the beam only touches the supports and surface effect can only be caused by actual contact; hence I conclude that the surface effect due to loading only is strictly represented by the hyperbola and is as if the beam were of infinite depth*.

In order to establish the hyperbolic law with greater certainty, experiments were made enabling as many as seven points on the curve to be obtained within 3·5 millim. of the point of contact, the highest point being about ·5 millim. from the top of the beam.

Within this range the effect due to the steel frame may with accuracy be neglected.

Experiment 2. A beam of annealed glass, 61 millim. × 6·5 millim. × 20 millim. deep, was placed in the steel straining-frame, on a piece of thick paper, and loaded as before with the steel roller 2 millim. in diameter.

Nicols crossed and at 45° to the axis of the beam.

The screw load was applied until six interference-fringes appeared under the roller; these were examined through a microscope with a micrometer-eyepiece divided to thousandths of an inch. Light from a sodium-flame was used, and the distance between the point of contact and the intersection of each fringe with the normal was measured in micrometer-divisions.

I. Distances in micrometer-divisions to successive fringes:

11·0 13·5 17·0 23·0 35·0 71·5,

but the shears are as 6 . 5 . 4 . 3 . 2 . 1, since there is a difference of phase of only ½ a wave-length required to produce a fringe, hence taking the products of distances into shears we get

66·0 67·5 68·0 69·0 70·0 71·5.

But we have so far neglected the value of θ, the distance of the axis of shears from the point of contact.

By taking the two most reliable observations, where the distance from the point of contact is large and yet where the fringes are well defined, we should have, if the law is hyperbolic,

---

* According to this reasoning there appears to be a shear of finite amount at the bottom of the beam—when doubly supported—due to loading only, but this does not seem to me to be inconsistent with the surface conditions.

$$3(23+\theta)=4(17+\theta),$$

or

$$\theta=1^*.$$

Correcting the original readings by adding θ to each, we get

12 14·5 18 24 36 72·5,

and the products become

72 72·5 72 72 72 72·5.

II. Same beam, &c., as before, roller and load readjusted. Distance to successive fringes :—

11·5 14·25 17·75 24·0 36·0 75·0

To find θ, take

$$3(24+\theta)=4(17{\cdot}75+\theta),\text{ or }\theta=1.$$

Correcting the distances, we have

12·5 15·25 18·75 25·0 37·0 76·0,

and the products of the distances into the shears become

75·0 76·25 75·0 75·0 74·0 76·0.

III. Same beam, &c., as before, roller and load readjusted. Distance to successive fringes :—

10·75 12·5 15·25 19·25 26·0 39·0 80·5.

To find θ take $3(26+\theta)=4(19{\cdot}25+\theta)$, whence $\theta=1$.

Correcting the distances, we have

11·75 13·5 16·25 20·25 27·0 40·0 81·5,

and the products become

82·25 81·0 81·25 81·0 81·0 80·0 81·5.

The law of variation of shear along the normal is thus hown to be hyperbolic.

---

* One micrometer-division = 0·044 millim.

Experiment 3. The steel straining-frame was removed from the instrument and—by a screw inserted in the place of the straining-screw—hung from a balance, which could be loaded with shot and had a leverage of 50 to 1 : a steel stirrup was hung over the frame with two hardened points resting on the two guiding-pins ; one lower end of the stirrup was secured to the body of the balance, the beam inserted and balanced, and shot put in the pan. This lifted the straining-frame and loaded the beam.

Beam [B] 56 millim. × 20 millim. × 6·5 millim. placed on the base of the steel frame on a piece of thin paper : loaded by a steel roller 2 millim. in diameter. Viewed through nicols crossed and at 45° to the horizontal axis of the beam.

The balance was loaded until the first blue fringe was brought down to a given position on the beam, and the weight of shot observed ; the same fringe was then brought down to a lower given position, and the weight of shot again observed, and so on for successive points.

Distance (α) of given points on normal from top of beam, in millim.	Load (β) on roller (weight of shot) × 50 in lb.			β/α.
	1.	2.	Mean.	
1·15	40	39	39·5	34·34
3·2	114	105	109·5	34·22
4·2	145	149	147	35·00
5·2	182	180	181	34·80
6·2	218	……	218	35·16

If the shear at 4·2 millim. with 147 lb. be taken as unity, the shear at 5·2 millim. with this same load will be $\frac{147}{181}$, since the same shear is produced at 5·2 millim. with 181 lb. as is produced at 4·2 millim. with 147 lb. Hence if the curve of loading is an hyperbola, we should have

$$4{\cdot}2 \times 1 = \frac{147}{181} \times 5{\cdot}2 \text{ or } \beta/\alpha \text{ a constant.}$$

From the third column given above the values of β/α will be seen to be nearly equal in each case ; the value of θ has here been neglected ; if we put $\theta = 0{\cdot}04$ millim., the values of β/α become

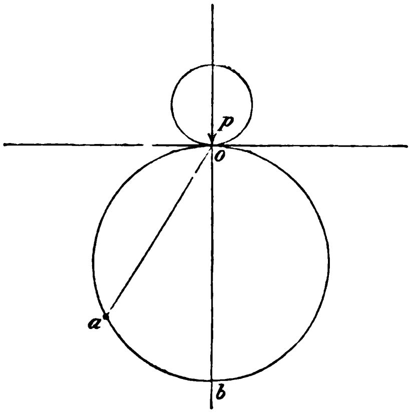
34·6 33·8 34·7 34·5 34·9.

Proposition II.

Things being arranged as in Proposition I., it is required to determine the locus of points of equal intensity of shear, and to show that at any point whatever the shear is inversely proportional to its distance from the point of contact.

Experiment 4. The beam was examined under circularly polarized light, as in Clerk-Maxwell's experiments, in order to obtain the variations in the amount of the strain uncomplicated by variations in the directions of the principal stress-axes; white light was used.

Fig. 2.



The loci of points of equal shear were found to be circles, as in the figure; circles of equal shear were obtained up to 8 millim. diameter with this beam.

Hence the shear at any point a equals the shear at b, if $o\,b\,a$ is a circle, and $o\,b$ the normal at o; *i.e.* shear at a

$$=k\frac{p}{ob}=k\frac{p\cos\theta}{o\,a},$$

k being some constant, but $p\cos\theta$ is the resolved part of the pressure at o* along $o\,a$; hence the shear at any point is inversely proportional to its distance from the point of contact.

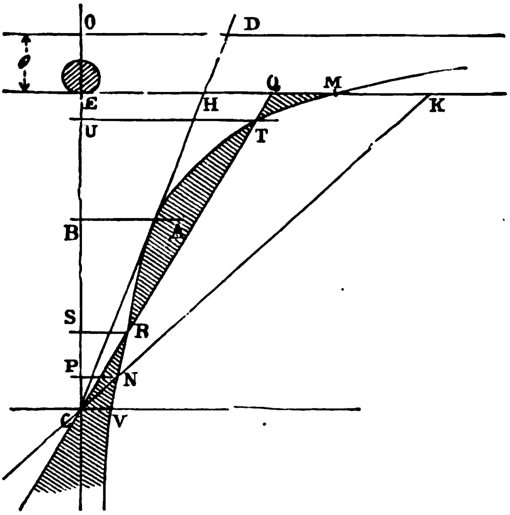
---

* See Professor Boussinesq's results quoted already.

Proposition III.

The state of strain at the centre of the beam when doubly supported may be found by superposing on the state of strain due to bending only, that due to surface-loading without bending.

Fig. 3.



It has been proved that the state of strain along the normal at the point of contact due to the surface-loading may be represented by an hyperbola whose asymptotes are respectively the normal itself and a line parallel to the axis of the beam at a distance θ from the point of contact. Let O C, O D in fig. 3 represent these asymptotes, O E $=\theta$; let an hyperbola be drawn whose ordinates parallel to O D represent the shear at any point along E C for a given load : since the shear is proportional to the compressive stress, these ordinates may be considered as proportional to the compressive stress at any point along E C.

By our (α) assumption we may represent the stresses at any point along E C, due to bending, by a right line drawn through C, the centre of the depth.

Let C K be such a line, drawn on the same scale as the hyperbola, so that E K represents the shear (vertical stretch) at E due to bending*, while E M represents the shear (vertical squeeze) due to loading.

These two curves must intersect at some point N; at the corresponding point P on the normal the shear (vertical squeeze) due to the loading is equal to the shear (vertical stretch) due to the bending: an element of volume at P will therefore be subject to voluminal compression only, and the shear will be zero, there will therefore be no birefringent action, and when viewed with crossed nicols there should be a dark spot on a white field.

If the load is kept constant and the span diminished, E K will decrease until C K cuts the hyperbola at a second point; we should now get two points of darkness. As the span is still diminished these dark points should rise and fall respectively until they coincide, when C K is a tangent to the hyperbola; after this they should separate out at right angles.

Plate V. fig. 2 gives the results of an experiment (5) made with constant load and varying spans. The beam was 128 millim. × 19 millim. deep × 5·5 millim. thick, supported on two steel rollers 2 millim. in diameter and centrally loaded over a similar roller: the nicols were crossed and at 45° to the axis. The following table gives the spans:—

Curve.	Span in millim.	Ratio of span to depth.
1	120	6·31
2	100	5·26
3	88	4·63
4	80	4·21
5	78	4·10
6	72	3·79

This experiment shows that there are, generally, two points of zero shear which close up as the span diminishes and then open at right angles.

---

* The compressive stress due to bending, at any point on C E, produces a shear (vertical stretch) and a voluminal compression, and both are proportional to the stress, similarly for the shear (vertical squeeze) and voluminal compression produced by the stress due to the loading; so for this purpose it is indifferent whether the ordinates of the two curves represent the compressive stresses or the shears produced.

The same phenomena may be observed by placing a beam on a flat surface and loading it, and then placing over this beam a second, which may be bent with a very long span, or by two couples at the end; the effect is the same for different degrees of bending as for varying spans in the former experiment.

Thus for spans of four to five depths the normal under the load is divided into three parts by two points of zero shear, elements between these points being subject to shear (vertical stretch), while elements above and below them are subject to shear (vertical squeeze).

When, however, the span is less than four depths, every element in the cross section under the load is subject to shear (vertical squeeze) and the greatest strained element is immediately under the load.

These results may be further checked and confirmed by examining each part of the normal by placing over it a beam bent in the hand; if the part under examination is in shear, say (vertical squeeze), darkness may be produced by superposing a part of the second beam oppositely strained; if the strains were similar, increased brightness would result.

I exhibit also the results of experiments made to determine the position of the black bands for lower ratios of span to depth.

The dimensions of the beam were 124 millim. × 20 millim. deep × 6·5 millim. thick, loaded on rollers like the others; nicols crossed and at 45° to the axis.

Here the effect of the supports is very marked, so that when $p=2$ the black band only just touches the axis.

It must be remembered that at the point where the black band cuts the normal the shear is zero, but that everywhere else on the band all that is indicated is that the directions of resultant tension and compression are at 45° to the axis of the beam.

Experiment 6 was made to establish Proposition III. with greater certainty.

Beam 128 millim. × 19 millim. × 5·5 millim. was placed on the base of the straining-frame, on a piece of thin paper and loaded with shot until the first blue fringe came down to a point 1·7 millim. from the top. The load was 65 lb.

The same beam was then supported on two steel rollers 2 millim. in diameter and 120 millim. apart, and centrally loaded over a similar roller until the same blue fringe appeared at the bottom of the beam. The load was 55 lb.

An hyperbola has been drawn (see fig. 1, Plate V.) of convenient proportions, cutting the horizontal through the above-mentioned point at 28·5 divisions from the normal; the shear corresponding to the blue fringe is thus represented by 28·5 divisions, and there is that shear at the point with a load of 65 lb.

Now the stress due to bending, at the extreme bottom fibre of a beam 19 millim. deep, 120 millim. span, and 5·5 millim. thick, with a load of 55 lb., is 1·436 tons per square inch.

The vertical compressive stress at this point, due to the load of 55 lb., is, as is shown later on, 0·121 ton per square inch; but we are not justified in superposing the shears produced by these two stresses, being tensile and compressive at right angles, and the former as much as twelve times the latter, so I shall take the stress at the blue fringe as 1·436 tons per square inch.

Hence the compressive stress produced by **65** lb. over a span of 120 miliim., at the top fibre, is

$$1{\cdot}436 \times \frac{65}{55},$$

and the corresponding value in scale-divisions is

$$1{\cdot}436 \times \frac{65}{55} \times \frac{28{\cdot}5}{1{\cdot}436} = 33{\cdot}7 \text{ divisions.}$$

This distance is set off along the top surface in the figure, and the point so found joined to the centre of the middle section: where it cuts the hyperbola we should get darkness on the normal with a span of 120 millim. We can also draw lines representing the bending-stresses for other spans for the same load of 65 lb.

The position of the black bands on the normal, as found by experiment for spans of 120 and 100 millim., are indicated on the normal, and will be found to agree very closely with those points found independently by the intersection of the two curves.

The curve of bending-stresses is a tangent to the curve of loading at a span of 73 millim., as measured from the figure, whereas it is apparently 82 millim. when actually observed ; it would appear more correct to determine this span by drawing the curve through two points which can be observed with accuracy, and then drawing the tangent and measuring the intercept, since the experimental determination of the span giving coincidence of the two dark bands is one liable to considerable error.

By drawing lines from the centre to the points along the top surface corresponding to longer spans we see that the deviation of the so-called "neutral axis" from the centre is considerable : thus even at a span = 10 depths = 190 millim. it should be 1 millim. above the centre.

Proposition IV.

The strain at every point along the normal due to loading varies directly as the load.

Experiment 7. The beam is placed on two supports as before, with a small central load, and the points of intersection of the black bands with the normal are noted. The load is now increased up to the safe limit when the points of intersection are observed to remain unaltered.

We know that the strain at any point on the normal due to bending is proportional to the load ; hence if the point of intersection of the curves of bending and loading remains the same when the load is increased, we know that the strain at any point due to the loading must vary as the load.

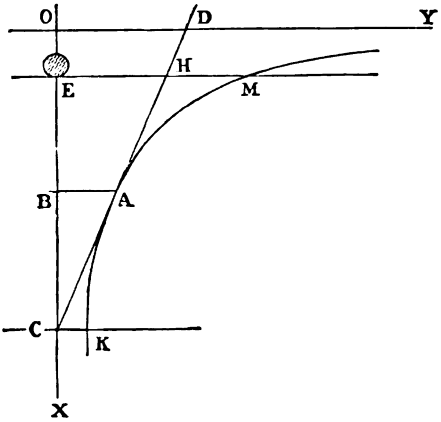
Proposition V.

To determine the constant in the equation to the curve of loading along the normal for any beam.

Let O X represent the vertical through the centre of a beam centrally loaded, E the point of contact of the load with the top of the beam E K ; O Y the axis of shear, O E $=\theta$; K A M the hyperbola of loading for any given load, C A H D the line of stresses due to bending along C E, for the same load, the span being chosen so that C A H is a tangent to the hyperbola at A; *i. e.* so that the dark bands coincide at B. Then O Y and O X are the asymptotes of the hyperbola.

It has been proved that the equation is of the form $y=k\frac{1}{x}$, where y is the compressive stress at a point on the normal E C at a distance X from O. If W is the load and b the width of the beam = length of bearing of loaded roller, we have

Fig. 4.



$$y=k.\frac{W}{bx}\text{—for the given beam.}$$

Then $\mathrm{OD}=\mathrm{BA}=2k\frac{W}{bx}$ (since B A represents the stress at B due to the load W). Also $\mathrm{EH}=\frac{3}{2}\frac{Wl}{h^2b}$, where E H represents the stress at E due to a load W on a beam of depth h and width b and span l; and

$$\frac{\mathrm{EH}}{\mathrm{OD}}=\frac{\mathrm{CE}}{\mathrm{CO}};\quad\therefore\ \mathrm{CO}=\mathrm{CE}\,\frac{\mathrm{OD}}{\mathrm{EH}},$$

$$\text{or }\frac{h}{2}+\theta=\frac{h}{2}\times 2k\frac{W}{bx}\times\frac{2}{3}\frac{h^2b}{Wl}=\frac{2}{3}\cdot\frac{h^3k}{lx}$$

$$= \frac{2}{3}\,\frac{h^3k}{\frac{l}{2}\left(\frac{h}{2}+\theta\right)} \quad [\text{since } x=\tfrac{1}{2}\text{CO}]\ ;$$

$$\therefore\ \left(\frac{h}{2}+\theta\right)^2 = \frac{4}{3}\,\frac{h^3k}{l}\ ;$$

$$\therefore\quad \frac{1}{4}+\frac{\theta}{h}+\frac{\theta^2}{h^2}=\frac{4}{3}\cdot\frac{h}{l}\cdot k\ ;\ \text{ put } \frac{l}{h}=\rho\ ;$$

$$\therefore\quad k=\frac{3}{4}\rho\left(\frac{1}{4}+\frac{\theta}{h}+\frac{\theta^2}{h^2}\right).$$

To find k the beam is placed on two supports and centrally loaded ; the two points where the black bands cross the normal are observed (the span being longer than four depths), and plotted, and an hyperbola drawn through them ; a tangent is then drawn to this curve from the centre of the section and its intercept on the upper edge measured, the span giving coincidence of the black bands can then be calculated.

Experiment 8. For a beam 128 millim. × 19 millim. deep × 5·5 millim. I find this span to be 73 millim. ; hence

$$\rho=\frac{73}{19}=3\cdot84.$$

Taking θ at 0·04 millim., $\frac{\theta}{h}=$ 0·002 millim., and neglecting $\frac{\theta^2}{h^2}$, we have

$$h=\tfrac{3}{4}\times 3\cdot84\times 0\cdot252=0\cdot726.$$

Proposition VI.

To verify the equation to the curve of loading.

Experiment 9. Beam 128 millim. × 19 millim. × 5·5 millim. The stress corresponding to the blue fringe with this beam was found, as already explained, by loading the beam over a span of 120 millim., until the blue fringe appeared at the bottom of the beam ; the load required was 55 lb. ; hence the corresponding stress is 1·436 tons per square inch*.

---

* From the equation $y=k\frac{W}{bx}$, there is a compressive stress of 0·121 ton per square inch here due to the load. I have not added the effect of this to that of the bending, as there is no proof that the superposition of small strains holds when the strains themselves are so unequal.

When laid on the base of the steel frame, the same fringe was observed at 1·7 millim. from the top with a load of 65 lb.

From the equation to the curve of loading, taking $k=0{\cdot}726$, $\theta=0{\cdot}04$ millim., we ought to have a stress at 1·7 millim. from the top equal to

$$y=0{\cdot}726\times\frac{25{\cdot}4^2}{2240}\times\frac{65}{1{\cdot}74}\times\frac{1}{5{\cdot}5}=1{\cdot}419 \text{ tons per square inch.}$$

The lines of Principal Stress afford a convenient means of studying the condition of strain in a bent beam.

In a memoir published in 1838* Lamé discussed the problem of the equilibrium of an elastic solid, and investigated the properties of what he termed "isostatic surfaces," or surfaces where only normal "actions" are applied.

In 1870 Saint-Venant† examined the differential equations to which the subject of "isostatic surfaces" gave rise, and in 1872 Professor Boussinesq‡ gave a geometric method for constructing isostatic lines passing through any given point. This memoir was shortly followed by a second§, treating of the integration of the equations involved.

Rankine has examined the form of the curves of Principal Stress, and given an expression from which the curves can be drawn‖. He neglects the surface-loading effect as "in most cases practically of small intensity when compared with the other elements of stress." On comparing his curves with those in Plate V. it will be noticed how closely the curves of tension agree, while the curves of compression are very dissimilar.

Sir George Airy has calculated and drawn the curves of principal stress for several cases of flexure, including that of a beam doubly supported and centrally loaded¶. He assumes "that there is a neutral point in the centre of the depth; that on the upper side of this neutral point the forces are forces of tension, and on the lower side are forces of compression, and that these forces are proportional to the distances from the neutral point;" but he says "These suppositions seem to imply that the actual extensions or compressions correspond exactly to the curvature of the edge of the lamina." The surface-loading effect is not here taken into account; and it would have been interesting to compare the results as shown in fig. 6, for a beam in which the span equals ten depths, with the actual curve as found by experiment. This comparison, however, would lead to erroneous conclusions, since it has been shown* that the results arrived at are not consistent with the fundamental equations, and the form of the curves can be accepted only as a very general approximation.

---

* *Comptes Rendus*, vol. vii. p. 778: "Mémoire sur les surfaces isostatiques dans les corps solides en équilibre d'élasticité."

† Ibid. vol. lxx. ‡ Ibid. vol. lxxiv. p. 242. § Ibid. vol. lxxiv. p. 318.

‖ 'Applied Mechanics,' §§ 310 and 311.

¶ Phil. Trans. 1863, part 1.

Proposition VII.

To determine the lines of Principal Stress in a glass beam doubly supported and centrally loaded.

Experiment 10.—A glass beam, 128 millim. × 19 millim. deep × 5·5 millim. thick, was placed in the steel straining-frame on two steel rollers 2 millim. in diameter, and centrally loaded over a similar steel roller.

The span chosen was 60 millim., giving for ρ the value 3·15.

The nicols were crossed and set at an observed angle, and the black band plotted on squared paper corresponding to the squares on the glass beam. This band of course represents the locus of points where the axes of principal stress are parallel to the directions of the planes of the nicol.

The nicols were then turned through a small angle α, the new position of the black bands plotted, and so on for several different angles. These curves are shown in Plate V. The lines of principal stress are easily deduced from these and are shown in Plate V. fig. 4.

Since communicating the above, Sir George Stokes has gone very fully into this problem, and has kindly allowed me to quote the following extracts from letters I have received from him on the subject:—

"Let A be the point in the upper surface where the pressure (P) is applied; B, C the points of support below, which I suppose to be equidistant from A; D the middle point of BC. Let y be measured downwards from A; denote BD or DC by a, and AB by b. You have the expressions for the stresses produced by P in an infinite solid $\left(x=\frac{2P}{\pi}\cdot\frac{1}{y}\right)$, and the question is, What system must we superpose on this to pass to the actual case? This, as I showed you, is the system of stresses produced by a system of forces applied to the surface. The forces consist—(1) of the two pressures $\frac{1}{2}$P at B and C; (2) of a continuous oblique tension below, represented in drawing by a fan of tensions directed at every point of the lower surface from the point A.

---

* See criticism on Sir George Airy's solution in Ibbetson's 'Mathematical Theory of Elasticity,' note on p. 358.

"Imagine now the beam cut into two by a plane along A D. Consider one half only, say that on the B side. Everything will remain the same as before, provided we supply to the surface A D forces representing the pressures or tensions which existed in the undivided beam. On account of the symmetry, the direction of these must be normal.

"At D the vertical pressure on a horizontal plane in the infinite solid is compounded with an equal vertical tension due to the fan. Hence, of the vertical pressure in A D which must be superposed on the vertical pressure in the infinite solid, we know thus much without obtaining a complete solusion of the problem, namely, that it must equal minus $2P/\pi b$ at D and 0 at A. If we suppose it to vary uniformly between, we are not likely to be far wrong.

"This leads to the following expression for the vertical pressure in A D:—

$$\frac{2P}{\pi}\left(\frac{1}{y}-\frac{y}{b^2}\right).$$

"Now for the horizontal. We know that the complete system of external forces must satisfy the conditions of equilibrium of a rigid body. The direction in each element of the fan passes through A, about which therefore the fan has no moment. Hence the moment of the horizontal forces along A D taken about A must equal $\frac{1}{2}$Pa. Again, the resultant of the semi-fan is a force passing through A, and its vertical component is $\frac{1}{2}$P. Its horizontal component is the integral of

$$\frac{2Pb^2}{\pi}\cdot\frac{x\,dx}{(b^2+x^2)^2},$$

taken from 0 to infinity, or $\frac{P}{\pi}$.

“Hence of the horizontal forces along A D we know these two things:—

(1) The sum must equal $\frac{P}{\pi}$,

(2) The moment round A must equal $\frac{1}{2}Pa$.

“In default of a knowledge of the law according to which the force varies with y, it is natural to take it, for a more or less close approximation, to be expressed by the linear function $A+By$, or say Y. To determine the arbitrary constants, A, B, we have only to equate the integral of $Y.dy$ to $\frac{P}{\pi}$, and that of $Yy.dy$ to $\frac{1}{2}Pa$, the limits being 0 to b. We thus get for the expression for the tension at any point of A D,

$$\frac{P}{b}\left(\frac{4}{\pi}-\frac{3a}{b}\right)+\frac{6P}{b}\left(\frac{a}{b}-\frac{1}{\pi}\right)\frac{y}{b}.$$

“At neutral points the vertical pressure equals minus the horizontal tension, giving

$$\left(\frac{6\pi a}{b}-8\right)\frac{y^2}{b^2}+\left(4-\frac{3\pi a}{b}\right)\frac{y}{b}+2=0\cdot$$

or, putting for shortness $\frac{3\pi a}{b}-4=m$,

$$2m\left(\frac{y}{b}\right)^2-m\frac{y}{b}+2=0,\quad\therefore\ \frac{y}{b}=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{m}}.$$

For the neutral points to be real and different, we must have

$$m>16,\quad \frac{2a}{b}>\frac{40}{3\pi}.$$

When the neutral points coalesce into one, we have m equal 16, y equal $\frac{b}{4}$; and for the ratio of the span to the depth,

$\frac{2a}{b}$ equal $\frac{40}{3\pi}$, equal 4·245, or, say, the span is 4¼ times the depth.

"As regards the horizontal tension at points along A D, you take a linear function of y as I do, and your condition of moments is the same as my (2), but in lieu of my (1) you do what is equivalent to taking the total tension *nil.* You further omit the correction to the vertical pressure when we pass from a solid of infinite depth to one terminated by a plane below. You further take the coefficient of $\frac{P}{y}$ as k, a constant to be determined by the observations, instead of $\frac{2}{\pi}$.

"Taking the place of the neutral point (at one fourth of the depth) and the ratio of span to depth as given by my formulæ, and then treating them as if they had been the results of experiment, and substituting in your formulæ for the determination of k, I got 0·7947 instead of 0·64. The largeness of your coefficient is I think fully accounted for by the employment of the formulæ which you used.

"In your method you take the stress belonging to the solid supposed infinitely deep, and superpose it on the stress corresponding to a pure bend.

"This comes to the same thing as retaining three terms only in the equation I gave in my letter for determining the y of the neutral points.

"The equation thus becomes

$$\frac{6\pi a}{b} \cdot \frac{y^2}{b^2} - \frac{3\pi a}{b} \cdot \frac{y}{b} + 2 = 0,$$

or

$$2m\frac{y^2}{b^2} - m\frac{y}{b} + 2 = 0,$$

where

$$m = \frac{3\pi a}{b} \text{ instead of } \frac{3\pi a}{b} - 4.$$

"When the two neutral points merge into one, we have in both cases alike y equal ¼ b, and the only difference is that $3\pi\frac{a}{b}$ equals m instead of m plus 4.

"If you had supposed the coefficient for the infinite solid to be an unknown quantity k, and had applied your observations to determine it, using my formulæ instead of your own, you would have got something very close indeed to 0·64.

"It is noteworthy that in your problem, taken as one in two dimensions, the theoretical stresses in the planes of displacement are independent of the ratio between the two elastic constants; in other words, independent of the value of Poisson's ratio."

I have calculated the positions of the neutral points from Sir George Stokes's formula

$$\frac{y}{b}=\frac{1}{4}\pm\sqrt{\frac{1}{16}-\frac{1}{m}}$$

for spans of 88, 100, and 120 millim. in a beam 128 millim. long × 5·5 millim. wide × 19 millim. deep. These are given in the following Table in the 2nd and 3rd columns. The results of actual observations (see p. 192) are given in columns 4 and 5; while columns 6 and 7 give the same points as found by plotting the intersection of the curves of pure bending and loading (infinite solid assumed):—

Span.	Distance of Neutral Points from top edge, by					
	Sir George Stokes's formula.		Observation.		Intersection of curves.	
88.........	6·3	3·2	6·4	3·3	6·9	2·7
100.........	7·0	2·5	7·2	2·5	7·3	2·3
120.........	7·7	1·8	7·8	1·8	7·8	1·75

The error by the intersection method is greater in proportion as the span is smaller, as might have been expected.

If the observed positions of the neutral points are inserted in Sir George Stokes's formula, the value 0·64 is obtained for the constant k in the equation $x=\frac{2P}{\pi}\cdot\frac{1}{y}$.

M'Gill University, Montreal,
October 12, 1891.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

JANUARY 1892.

XXVII. *Table of Zonal Spherical Harmonics, calculated by* Messrs. C. E. Holland, P. R. Jones, *and* C. G. Lamb *With a short Explanation and some Illustrations of its use by* Professor JOHN PERRY, *D.Sc., F.R.S.**

[Plate VI.]

I HAD intended merely to present to the Society for publication a table of Zonal Spherical Harmonics. But some Members whom I have consulted seem to think that I ought also to give one or two examples of the practical use of such a table. The Members of the Society will, I hope, pardon my putting before them one or two elementary examples.

The use of Spherical Harmonics in the numerical solution of practical problems is almost unknown, I believe, except at the Finsbury Technical College, where, every year, I have been accustomed to make some Electrical Engineering students work a few common examples. My students have for some years been in possession of tables of zonal harmonics, but this year I have thought it well to make the tables more complete and to get them published for the general use of students of practical physics.

I have been told that many of the users of such a table would be glad of a few statements of the general principles underlying its use. For the proofs of these statements readers are referred to Mathematical treatises. Many readers will be satisfied with the treatment of the subject in Mr. Ferrer's excellent treatise, which is, however, written only for beginners.

---

* Read November 14, 1890.

In problems on Heat Conduction (V being temperature), on Hydrodynamics of incompressible fluids (V being velocity-potential*), in Electrostatics (V being electric potential), in Magnetism (V being magnetic potential), and in many other applications of Physics, we require to find V a function of x, y, z which shall satisfy the equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, as it is usually written, $\nabla^2V=0$ . . . (1), and which shall also satisfy certain other conditions. Now there are many kinds of function which satisfy equation (1). The definition of a Spherical Harmonic is "a *homogeneous* function of x, y, and z, which satisfies equation (1)."

If such a function can be found, say of the ith degree, and if we divide it by r^{2i+1} where $r^2=x^2+y^2+z^2$, it can be proved that the resulting expression will also satisfy (1), where i may be a positive or negative integer or fraction.

Now if a Spherical Harmonic of degree i (generally called a Solid Spherical Harmonic) be divided by r^i, we get what is called a *Surface* Spherical Harmonic of degree i.

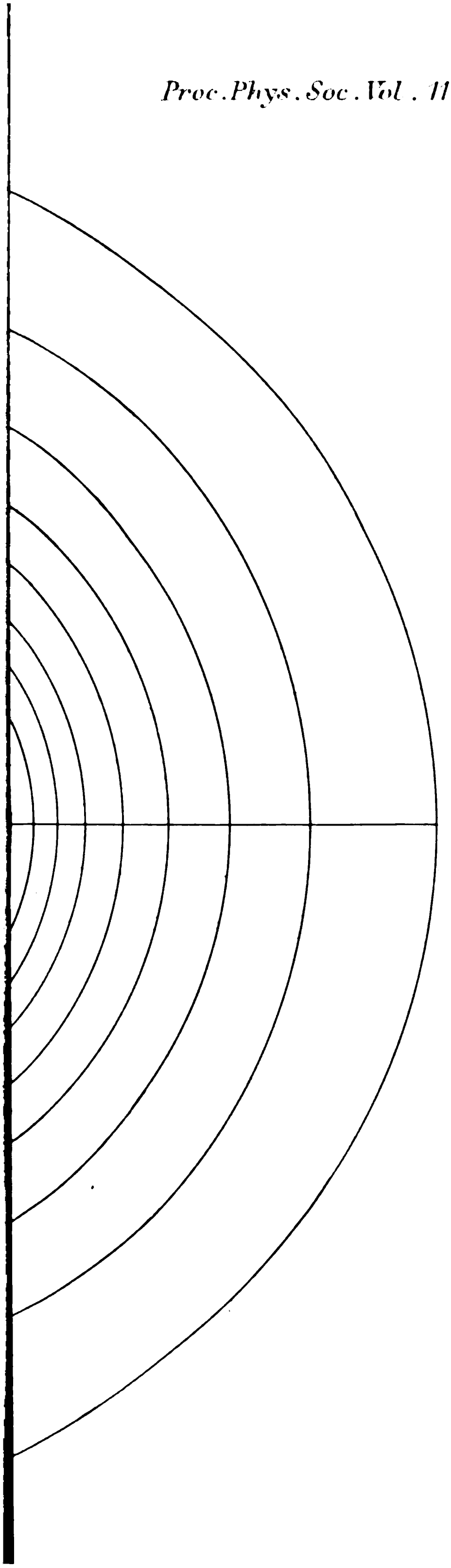
It was shown by Green that if there is a function V which satisfies equation (1) at every point of any given surface, then it is the only function which satisfies (1) throughout space; and there is always a function V obtainable which satisfies (1). It is the characteristic property of a surface spherical harmonic distribution of density of attracting matter on a spherical surface, that it produces a similar and similarly placed spherical harmonic distribution of potential

---

* When there exists a velocity-potential V in a portion of fluid, we mean that the velocity of the fluid at any place resolved in the direction s is

$$-\frac{dV}{ds}.$$

When the motion is "rotational," as in the wheel of a centrifugal pump or turbine, a velocity-potential does not exist. In any portion of a frictionless fluid, if there is irrotationality, that is, if there is a velocity-potential, the property cannot be destroyed.



Mintern Bros. lith.

over any concentric spherical surface throughout space, external and internal.

Instead of using x, y, and z coordinates we may of course use r, θ, and ϕ coordinates.

In a great number of practical cases V is symmetrical about an axis, and a symmetrical spherical harmonic is said to be a Zonal Spherical Harmonic. Taking the axis of symmetry as the axis of z, V is a function of z and $\sqrt{x^2+y^2}$. Or, in polar coordinates, V is a function of r and θ.

Let O be a point in the axis, the origin of coordinates; let the distance of any point P from O be called r, let the angle between O P and the axis be called θ, then in any distribution which has an axis of symmetry we need only to know r and θ. And over any spherical surface whose centre is O, the distribution will be a function of θ. Any zonal surface spherical harmonic is then merely a function of θ, and I give a table of values of these for values of θ differing by 1° from 0° to 90°, up to the harmonic of the seventh degree. These are indicated by P_0, P_1, P_2, &c. P_7.

The surface harmonic of no degree is 1, and is indicated by P_0.

The student is referred to Mathematical treatises for the proof that, if μ be written to represent cos θ, then

$$P_0=1,$$

$$P_1=\mu,$$

$$P_2=\frac{3\mu^2-1}{2},$$

$$P_3=\frac{5\mu^3-3\mu}{2},$$

$$P_4=\frac{35\mu^4-30\mu^2+3}{8},$$

$$P_5=\frac{63\mu^5-70\mu^3+15\mu}{8},$$

$$P_6=\frac{231\mu^6-315\mu^4+105\mu^2-5}{16},$$

$$P_7=\frac{429\mu^7-693\mu^5+315\mu^3-35\mu}{16}.$$

Any function of θ may be expanded in terms of P_0, P_1, P_2, &c., that is any symmetrical function V may be expanded in a series of Zonal Spherical Harmonics. Take, for example, the powers of cos θ, it may be shown that

$$\mu^0 = 1 = P_0,$$

$$\mu = P_1,$$

$$\mu^2 = \frac{2}{3} P_2 + \frac{1}{3} P_0,$$

$$\mu^3 = \frac{2}{5} P_3 + \frac{3}{5} P_1,$$

$$\mu^4 = \frac{8}{35} P_4 + \frac{4}{7} P_2 + \frac{1}{5} P_0,$$

$$\mu^5 = \frac{8}{63} P_5 + \frac{4}{9} P_3 + \frac{3}{7} P_1.$$

And also

$$\text{Cos } \theta = P_1,$$

$$\text{Cos } 2\theta = \frac{4}{3} P_2 - \frac{1}{3} P_0,$$

$$\text{Cos } 3\theta = \frac{8}{5} P_3 - \frac{3}{5} P_1,$$

$$\text{Cos } 4\theta = \frac{64}{35} P_4 - \frac{16}{21} P_2 - \frac{1}{15} P_0,$$

$$\text{Cos } 5\theta = \frac{128}{63} P_5 - \frac{8}{9} P_3 - \frac{1}{7} P.$$

There is an easy rule for expanding any function of θ in terms of P_0, P_1, P_2, &c.

The following table up to P_5 was primarily calculated separately by Messrs. C. E. Holland and P. R. Jones, who checked each other's result by comparison.

Mr. C. G. Lamb applied certain checks to the results of Messrs. Holland and Jones, and then calculated P_6 and P_7.

It is quite easy to extend the table to P_8, for it can be proved that there is a law connecting three consecutive harmonics, say the $(n-2)$th, $(n-1)$th, and the nth.

$$nP_n = (2n-1)\mu P_{n-1} - (n-1) P_{n-2}.$$

θ.	P_1.	P_2.	P_3.	P_4.	P_5.	P_6.	P_7.
0°	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1	·9998	·9995	·9991	·9985	·9977	·9967	·9955
2	·9994	·9982	·9963	·9939	·9909	·9872	·9829
3	·9986	·9959	·9918	·9863	·9795	·9713	·9617
4	·9976	·9927	·9854	·9758	·9638	·9495	·9329
5	·9962	·9886	·9773	·9623	·9437	·9216	·8961
6	·9945	·9836	·9674	·9459	·9194	·8881	·8522
7	·9925	·9777	·9557	·9267	·8911	·8476	·7986
8	·9903	·9709	·9423	·9048	·8589	·8053	·7448
9	·9877	·9633	·9273	·8803	·8232	·7571	·6831
10	·9848	·9548	9106	·8532	·7840	·7045	·6164
11	·9816	·9454	·8923	·8238	·7417	·6483	·5461
12	·9781	·9352	·8724	·7920	·6966	·5892	·4732
13	·9744	·9241	·8511	·7582	·6489	·5273	·3940
14	·9703	·9122	·8283	·7224	·5990	·4635	·3219
15	·9659	·8995	·8042	·6847	·5471	·3982	·2454
16	·9613	·8860	·7787	·6454	·4937	·3322	·1699
17	·9563	·8718	·7519	·6046	·4391	·2660	·0961
18	·9511	·8568	·7240	·5624	·3836	·2002	·0289
19	·9455	·8410	·6950	·5192	·3276	·1347	−·0143
20	·9397	·8245	·6649	·4750	·2715	·0719	−·1072
21	·9336	·8074	·6338	·4300	·2156	·0107	−·1662
22	·9272	·7895	·6019	·3845	·1602	−·0481	−·2201
23	·9205	·7710	·5692	·3386	·1057	−·1038	−·2681
24	·9135	·7518	·5357	·2926	·0525	−·1559	−·3095
25	·9063	·7321	·5016	·2465	·0009	−·2053	−·3463
26	·8988	·7117	·4670	·2007	−·0489	−·2478	−·3717
27	·8910	·6908	·4319	·1553	−·0964	−·2869	−·3921
28	·8829	·6694	·3964	·1105	−·1415	−·3211	−·4052
29	·8746	·6474	·3607	·0665	−·1839	−·3503	−·4114
30	·8660	·6250	·3248	·0234	−·2233	−·3740	−·4101
31	·8572	·6021	·2887	−·0185	−·2595	−·3924	−·4022
32	·8480	·5788	·2527	−·0591	−·2923	−·4052	−·3876
33	·8387	·5551	·2167	−·0982	−·3216	−·4126	−·3670
34	·8290	·5310	·1809	−·1357	−·3473	−·4148	− 3409
35	·8192	·5065	·1454	−·1714	−·3691	−·4115	−·3096
36	·8090	·4818	·1102	−·2052	−·3871	−·4031	−·2738
37	·7986	·4567	·0755	−·2370	−·4011	−·3898	−·2343
38	·7880	·4314	·0413	−·2666	−·4112	−·3719	−·1918
39	·7771	·4059	·0077	−·2940	−·4174	−·3497	−·1469
40	·7660	·3802	−·0252	−·3190	−·4197	−·3234	−·1003
41	·7547	·3544	−·0574	−·3416	−·4181	−·2938	−·0534
42	·7431	·3284	−·0887	−·3616	−·4128	−·2611	−·0065
43	·7314	·3023	−·1191	−·3791	−·4038	−·2255	·0398
44	·7193	·2762	−·1485	−·3940	−·3914	−·1878	·0846
45	·7071	·2500	−·1768	− 4062	−·3757	−·1485	·1270
46	·6947	·2238	−·2040	−·4158	−·3568	−·1079	·1666
47	·6820	·1977	−·2300	−·4252	−·3350	−·0645	·2054
48	·6691	·1716	−·2547	−·4270	−·3105	−·0251	·2349
49	·6561	·1456	−·2781	−·4286	−·2836	+·0161	·2627
50	·6428	·1198	−·3002	−·4275	−·2545	+·0563	·2854
51	·6293	·0941	−·3209	−·4239	−·2235	+·0954	·3031
52	·6157	·0686	−·3401	−·4178	−·1910	+·1326	·3153
53	·6018	·0433	−·3578	−·4093	−·1571	+·1677	·3221
54	·5878	·0182	−·3740	−·3984	−·1223	+·2002	·3234

Table (*continued*).

θ.	P_1.	P_2.	P_3.	P_4.	P_5.	P_6.	P_7.
55°	·5736	−·0065	−·3886	−·3852	−·0868	+·2297	·3191
56	·5592	−·0310	−·4016	−·3698	−·0510	+·2559	·3095
57	·5446	−·0551	−·4131	−·3524	−·0150	+·2787	·2949
58	·5299	−·0788	−·4229	−·3331	·0206	+·2976	·2752
59	·5150	−·1021	−·4310	−·3119	·0557	+·3125	·2511
60	·5000	−·1250	−·4375	−·2891	·0898	+·3232	·2231
61	·4848	−·1474	−·4423	−·2647	·1229	+·3298	·1916
62	·4695	−·1694	−·4455	−·2390	·1545	+·3321	·1571
63	·4540	−·1908	−·4471	−·2121	·1844	+·3302	·1203
64	·4384	−·2117	−·4470	−·1841	·2123	+·3240	·0818
65	·4226	−·2321	−·4452	−·1552	·2381	+·3138	·0422
66	·4067	−·2518	−·4419	−·1256	·2615	+·2996	·0021
67	·3907	−·2710	−·4370	−·0955	·2824	+·2819	−·0375
68	·3746	−·2896	−·4305	−·0650	·3005	+·2605	−·0763
69	·3584	−·3074	−·4225	−·0344	·3158	+·2361	−·1135
70	·3420	−·3245	−·4130	·0038	·3281	+·2089	−·1485
71	·3256	−·3410	−·4021	·0267	·3373	+·1786	−·1811
72	·3090	−·3568	−·3898	·0568	·3434	+·1472	−·2099
73	·2924	−·3718	−·3761	·0864	·3463	+·1144	−·2347
74	·2756	−·3860	−·3611	·1153	·3461	+·0795	−·2559
75	·2588	−·3995	−·3449	·1434	·3427	+·0431	−·2730
76	·2419	−·4112	−·3275	·1705	·3362	+·0076	−·2848
77	·2250	−·4241	−·3090	·1964	·3267	−·0284	−·2919
78	·2079	−·4352	−·2894	·2211	·3143	−·0644	−·2943
79	·1908	−·4454	−·2688	·2443	·2990	−·0989	−·2913
80	·1736	−·4548	−·2474	·2659	·2810	−·1321	−·2835
81	·1564	−·4633	−·2251	·2859	·2606	−·1635	−·2709
82	·1392	−·4709	−·2020	·3040	·2378	−·1926	−·2536
83	·1219	−·4777	−·1783	·3203	·2129	−·2193	−·2321
84	·1045	−·4836	−·1539	·3345	·1861	−·2431	−·2067
85	·0872	−·4886	−·1291	·3468	·1577	−·2638	−·1779
86	·0698	−·4927	−·1038	·3569	·1278	−·2811	−·1460
87	·0523	−·4959	−·0781	·3648	·0969	−·2947	−·1117
88	·0349	−·4982	−·0522	·3704	·0651	−·3045	−·0735
89	·0175	−·4995	−·0262	·3739	·0327	−·3105	−·0381
90	·0000	−·5000	−·0000	·3750	·0000	−·3125	·0000

N.B.—To find P_i, if θ lies between 90° and 180°, look up the value for the angle $180° - \theta$, and change the sign if i be odd.

When P_1, P_2, &c. are plotted as radial heights and depths above and below a quadrant, the resulting curves, especially if coloured, are exceedingly interesting. A student can draw them to a useful scale in about one hour.

Example I.

The density σ of attracting matter on a spherical shell

1 centim. in radius is proportional to the square of the distance of any point from a diametral plane, being 6 per square centimetre where greatest: find the potential A inside and B outside. Taking the diametral plane as the equator and θ as the co-latitude, it is obvious that

$$\sigma = 6\mu^2.$$

The expansion of μ^2 in spherical harmonics is already given as

$$\mu^2 = \frac{1}{3} P_0 + \frac{2}{3} P_2.$$

So that we have σ in spherical harmonics,

$$\sigma = 2P_0 + 4P_2.$$

Hence, as A and B are derivable from the same surface harmonics,

$$A = A_0 P_0 + A_2 r^2 P_2,$$

$$B = \frac{B_0}{r} P_0 + \frac{B_2}{r^3} P_2;$$

where A_0, A_2, B_0, B_2 are constants to be found.

Now at the surface, that is where $r=1$, $A=B$, and we can apply this to every harmonic separately. Hence

$$A_0 = B_0, \quad A_2 = B_2.$$

Again, we know from the theory of attraction that the resultant force just outside and just inside the shell differs by the amount $4\pi\sigma$, or

$$\frac{dA}{dr} - \frac{dB}{dr} = 4\pi\sigma\ ;$$

and this is to be applied to every harmonic separately. Thus, taking terms involving P_0, we have

$$0 + B_0 r^{-2} = 4\pi \times 2\ ;$$

or putting $r=1$,

$$B_0 = 8\pi.$$

Again, taking the second terms,

$$2A_2 r + 3B_2 r^{-4} = 4\pi \times 4\ ;$$

or, as $A_2 = B_2$, and putting $r=1$,

$$5B_2 = 16\pi, \quad \therefore\ B_2 = \frac{16}{5}\pi.$$

Hence we have

$$\text{Inside potential} \quad A = 8\pi P_0 + \frac{16}{5}\pi r^2 P_2,$$

$$\text{Outside potential} \quad B = \frac{8\pi}{r} P_0 + \frac{16}{5}\pi \frac{1}{r^3} P_2\,;$$

or

$$\frac{A}{8\pi} = P_0 + \frac{2}{5} r^2 P_2 = \alpha \text{ say}, \quad \ldots\ldots \quad (1)$$

$$\frac{B}{8\pi} = \frac{1}{r} P_0 + \frac{2}{5}\frac{1}{r^3} P_2 = \beta \text{ say}. \quad \ldots \quad (2)$$

As, for our purpose, the actual unit of potential is unimportant, we will use α for $\frac{A}{8\pi}$, and β for $\frac{B}{8\pi}$.

I had no notion of how the equipotential surfaces would shape, and I tried to avoid forming any such notion, as it was my object to test the usefulness of the tables in working out any new problem. The first thing that it strikes one to do in this case is to find the potential on the sphere itself. This can be done from either (1) or (2) by putting $r=1$, and then

$$\alpha = P_0 + \frac{2}{5} P_2, \quad \text{or } 1 + \frac{2}{5} P_2.$$

We can now find α for various values of θ, using the table. Thus when $\theta = 0$, $P_2 = 1$, and therefore $\alpha = 1{\cdot}4$. Thus we have the following values:—

θ	0	15	30	45	60	75	90
P_2......	1	·9	·625	·25	−·125	−·3995	−·5000
α	1·4	1·36	1·25	1·1	0·95	0·84	0·80

Next, take any value of θ, say $\theta = 45°$. Then from the table, $P_2 = 0{\cdot}2500$, so that

$$\text{inside,} \quad \alpha = 1 + \frac{1}{10} r^2\,;$$

$$\text{outside, } \beta = \frac{1}{r} + \frac{1}{10}\frac{1}{r^3}, \quad \text{or } \frac{10r^2+1}{10r^3}.$$

For any value of r less than 1 we calculate α; for any value of r greater than 1 we calculate β.

For $\theta = 45°$.

r...	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
α...	1	1·001	1·004	1·009	1·016	1·025	1·036	1·049	1·064	1·081	1·100
r...	1·2	1·4	1·6	1·8	2·0	2·2	2·6	3·0	3·5	4	5
β...	·8912	·7507	·65	·572	·512	...	...	...	...	...	·2

Now on a sheet of squared paper I plotted the values of r and α or β, and so found the values of r for such particular values of α or β as seemed suitable for curve-drawing. In fact I found the values of r for $\theta = 45°$ for various equipotential surfaces.

Repeating this for other values of θ and drawing radial lines on a sheet of paper, it was easy to draw the equipotential surfaces.

The figure (Pl. VI.) was obtained in this way by Mr. Joselin. It shows the equipotential surfaces from 0·5 to 1·4. These surfaces are surfaces of revolution. Of course the resultant force anywhere is inversely as the normal distance apart of the equipotential surfaces, and the direction of the force is everywhere normal to the equipotential surfaces.

Example II.

One circular spire of wire of radius a has an electric current C flowing in it. Find the electromagnetic potential everywhere.

At any point on the axis, z centimetres from the centre, the potential is

$$V = 2\pi C\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right). \quad \ldots \quad (1)$$

It is always worth while at first to get an idea of the range of values of V.

Putting $z=0, \quad V=2\pi C.$

$z=a, \quad V=\cdot 293\times 2\pi C.$

$z=\infty, \quad V=0.$

For the sake of ease of calculation let $2\pi C=1$, and let $a=1$ centim.

Then
$$V=1-\frac{z}{\sqrt{z^2+1}}.$$

We can expand this in powers both of z and of $\frac{1}{z}$, and we have either

$$V=1-z+\frac{1}{2}z^3-\frac{3}{8}z^5+\frac{5}{16}z^7-\frac{35}{128}z^9+\&c. \quad . \quad . \quad (2)$$

or

$$V=\frac{1}{2}\frac{1}{z^2}-\frac{3}{8}\frac{1}{z^4}+\frac{5}{16}\frac{1}{z^6}-\frac{35}{128}\frac{1}{z^8}+\&c. \quad . \quad . \quad . \quad (3)$$

Now, if we can find V as a function of r and θ which is correct along the axial line, then it must be correct everywhere. [This is Green's theorem, assuming that the axial line to infinity is a cylindric surface of no lateral dimensions.] But it is obvious that

$$V=1-rP_1+\frac{1}{2}r^3P_3-\frac{3}{8}r^5P_5+\frac{5}{16}r^7P_7-\frac{35}{128}r^9P_9+\&c. \quad . \quad (4)$$

becomes (2) when $\theta=0$; and

$$V=\frac{1}{2}\frac{P_1}{r^2}-\frac{3}{8}\frac{P_3}{r^4}+\frac{5}{16}\frac{P_5}{r^6}-\frac{35}{128}\frac{P_7}{r^8}+\&c. \quad . \quad . \quad (5)$$

becomes (3) when $\theta=0$.

Hence these express the potential everywhere. The first of these is useful for calculation only when r is less than 1. The second is useful only when r is greater than 1.

And this shows a defect of the Spherical Harmonic method. For if r is nearly 1 we cannot easily calculate V from either of the series, having to use too many terms. It will, however, be found, even in this case, that when we use the harmonics up to P_{11} for any value of θ we can plot V on squared paper from $r=0$ to $r=0{\cdot}9$, and from $r=1{\cdot}1$ to values of r as large as we please. If the intermediate part of the curve from $r=0{\cdot}9$ to $r=1{\cdot}1$ be drawn with a little judgment, it is astonishing how quickly and accurately the equipotential surfaces may be drawn. The result may be compared with the lines of force as worked out by the elliptic integral method of Sir William Thomson.

Any such Electromagnetic solution is also the solution of a Hydrodynamic problem.

The principle adopted in this example is very useful. It is this:—If the potential at any point along an axis is expressible as

$A_0 + A_1 z + A_2 z^2 +$ &c., where A_0, A_1, A_2, &c. are constants,

then the potential *anywhere* is

$$A_0 P_0 + A_1 P_1 r + A_2 P_2 r^2 + \text{\&c.}$$

If the potential along an axis is

$\frac{B_0}{z} + \frac{B_1}{z^2} + \frac{B_2}{z^3} +$ &c., where B_0, B_1, B_2, &c. are constants,

the potential anywhere is

$$\frac{B_0 P_0}{r} + \frac{B_1 P_1}{r^2} + \frac{B_2 P_2}{r^3} + \text{\&c.}$$

I gave to Mr. Holland, as an example, the case of a hollow cylindric coil of wire, 2 centim. long, 1 centim. in radius inside, and 2 centim. radius outside: to find the magnetic potential everywhere when there are n turns of wire in the coil per unit length of the coil and there is unit current in the wire. Mascart gives for such a case the force at any point of the axis, and it is possible to expand Mascart's expression in powers of z and again in powers of $\frac{1}{z}$; so that it is easy to get the potential in powers of z and of $\frac{1}{z}$, and therefore, as in the last case, the potential everywhere. Now inside the coil and well outside it only a few terms of the series need calculation; but just at the ends of the coil the calculation is troublesome because many terms are required.

Mr. Holland tried to shorten the work in the following way. He found that the potential at points along the axis could be expressed *approximately* from $z=0$ to $z=2{\cdot}5$, with a maximum error anywhere of only 2 per cent., by

$$V=2\pi n^2(2-{\cdot}4884z+{\cdot}05513z^3-{\cdot}00518z^5+{\cdot}00022z^7),$$

z being measured from the middle point of the coil.

Hence, he said, the magnetic potential at a point r, θ is

$$V=2\pi n^2\{2-{\cdot}4884\,P_1r+{\cdot}05513\,P_3r^3-{\cdot}00518\,P_5r^5+{\cdot}00022\,P_7r^7\}.$$

From this he plotted equipotential surfaces and lines of force. He found that, inside the coil, and indeed everywhere near the coil except certain critical positions, these were approximately correct. But at the flat ends close to the wire they were absurdly wrong. It is easy now to see the physical meaning of Mr. Holland's approximation, and why we cannot use this ingenious quick way of working; but we had to be taught by experience.

Example III.

A solid bounded by a surface of revolution moves axially in an infinite mass of incompressible fluid which has no other motion than this gives to it. Find the motion.

In this case

$$\nabla^2 V=0:$$

and $-\frac{dV}{dn}$ at any point at the surface of the solid (dn being an element of the normal) in the fluid is really the normal velocity of the solid itself. Again, V must be constant at any infinite distance.

This may easily be worked out for any surface of revolution. Applying it to a sphere of 1 centim. radius, let v be the velocity of the centre of the sphere in the direction parallel to the axis of z. Then $\frac{dV}{dr}=v\cos\theta$ at any point at the surface of the sphere.

Expressing V in zonal harmonics, and taking it as 0 at an infinite distance,

$$V=\frac{A_0P_0}{r}+\frac{A_1P_1}{r^2}+\frac{A_2P_2}{r}+\&c.,$$

$\frac{dV}{dr}$ when $r=1$ is

$$v\cos\theta=-A_0P_0-2A_1P_1-3A_2P_2-\&c.$$

But $P_1=\mu=\cos\theta$, so that the other coefficients vanish, and

$$A=-\frac{v}{2},$$

so that

$$V=-\tfrac{1}{2}\frac{v}{r^2}P_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equipotential surfaces which a student will draw from (1) are perfectly well known. It is a good exercise to draw them.

If now a velocity $-v$ is impressed upon the whole system, sphere and fluid, we have the case of the sphere at rest and the fluid moving past it. We now merely add the term $-vr\cos\theta$ to (1) and obtain

$$V=-v\left(r+\frac{1}{2r^2}\right)P_1, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which again is easily represented upon paper in equipotential surfaces.

XXVIII. *On the Generalizations of* Van der Waals *regarding "Corresponding" Temperatures, Pressures, and Volumes. By* SYDNEY YOUNG, *D.Sc., Professor of Chemistry, University College, Bristol*[*].

[Plates VII.–IX.]

IN his dissertation *Die Continuität des gasformigen und flüssigen Zustandes* (Roth's translation, Leipzig, 1881, p. 128) Van der Waals has deduced the following generalizations from his fundamental equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=R(1+\alpha t).$$

If the absolute temperatures of various substances be proportional to their absolute critical temperatures, their vapour-pressures will be proportional to their critical pressures, and their volumes, both as liquid and as saturated vapour, will be proportional to their critical volumes.

---

[*] Read November 6, 1891.

At the time, however, that Van der Waals's great work was published, the available experimental data were insufficient to satisfactorily test the accuracy of these generalizations.

Since then the conclusions of Van der Waals have been subjected to a large amount of criticism, both adverse and favourable; and while it is now generally recognized that the relations between the temperatures, pressures, and volumes of liquids and gases cannot be represented by so simple a formula as that quoted above, it has nevertheless been accepted as correct by some authors, who have made it the basis of further generalizations.

And although the general verdict with regard to the strict accuracy of the formula itself can hardly be considered favourable, the notion of "corresponding" states has received wide acceptance; and, indeed, the generalizations regarding corresponding temperatures, pressures, and volumes might still be true even though the formula on which they were originally based required some alteration.

In order to study the relations, for instance, between the specific volumes of different substances, determinations were made in the first place at the same temperature, generally at 0° C.; later on it was considered that the conditions would be more uniform if the comparison were made at the boiling-points of the substances under normal atmospheric pressure. It is now, however, usually admitted that in order to obtain the best results the volumes should be determined at "corresponding" temperatures—that is to say, at absolute temperatures proportional to the absolute critical temperatures of the various bodies—or at their boiling-points under corresponding pressures—the two methods of comparison being, according to Van der Waals, identical.

During the last four years I have been engaged in a research on the vapour-pressures and specific volumes—both in the liquid state and as saturated vapour—of the following substances:—benzene, fluorbenzene, chlorobenzene, bromobenzene, iodobenzene*, carbon tetrachloride and stannic chloride†;

---

* Trans. Chem. Soc. lv. p. 486, and lix. p. 125.

† Ibid. lix. p. 911.

the vapour-pressures and specific volumes of methyl*, ethyl†, and propyl alcohols‡, and of ethyl ether§ have also been determined by Dr. Ramsay and myself. We obtained the same constants for acetic acid‖ up to 280°; and I have recently extended the observations with this substance up to the critical point (ibid. lix. p. 903).

The methods employed for the determination of the vapour-pressures of benzene and its halogen derivatives and for the specific volumes of these bodies in the liquid state were, with slight modifications, the same as those made use of by Ramsay and myself in our researches on the alcohols and ether.

Carbon tetrachloride, however, acts on mercury at high temperatures, and stannic chloride renders it unfit for use—though the chemical action is very slight—even at low temperatures; a considerable alteration in the method of determining vapour-pressures was therefore made (Trans. Chem. Soc. lix. p. 917).

For the determination of the specific volumes of stannic chloride in the liquid state and of all the substances in the condition of saturated vapour an entirely different method was employed, and this has been fully described in the Trans. Chem. Soc. 1891, p. 37. This method is applicable to substances that attack mercury, and from the data it affords the specific volumes both of liquid and saturated vapour may be calculated; it is also available at any temperature up to the critical point. A modification of the method is described in the Trans. Chem. Soc. lix. p. 929.

The object of this paper is to show how far the generalizations of Van der Waals have been verified by the experimental results.

If all the relations were strictly true, it would obviously make no difference whether the specific (or molecular) volumes were compared at corresponding temperatures or corresponding pressures; but it may be stated at once that it is only in a limited number of cases that, when the absolute temperatures are proportional to the absolute critical temperatures, the vapour-pressures are also proportional to the critical pressures. It is therefore necessary to compare the various substances, not only at corresponding temperatures but also at corresponding pressures. The best mode of procedure would probably be to give the temperatures, pressures, and volumes of each substance in terms of the critical constants of that body; but the critical volumes of only a few of the compounds have been directly determined, and it has therefore been necessary to compare the various substances with one of them taken as a standard. The very simple relations observed between the four monohalogen derivatives of benzene (Trans. Chem. Soc. 1889, p. 486), and the fact that the constants of fluorbenzene have been determined up to the critical point, render that substance the most suitable for the purpose.

---

* Phil. Trans. clxxviii. A, p. 313.

† Ibid. 1886, part i. p. 123.

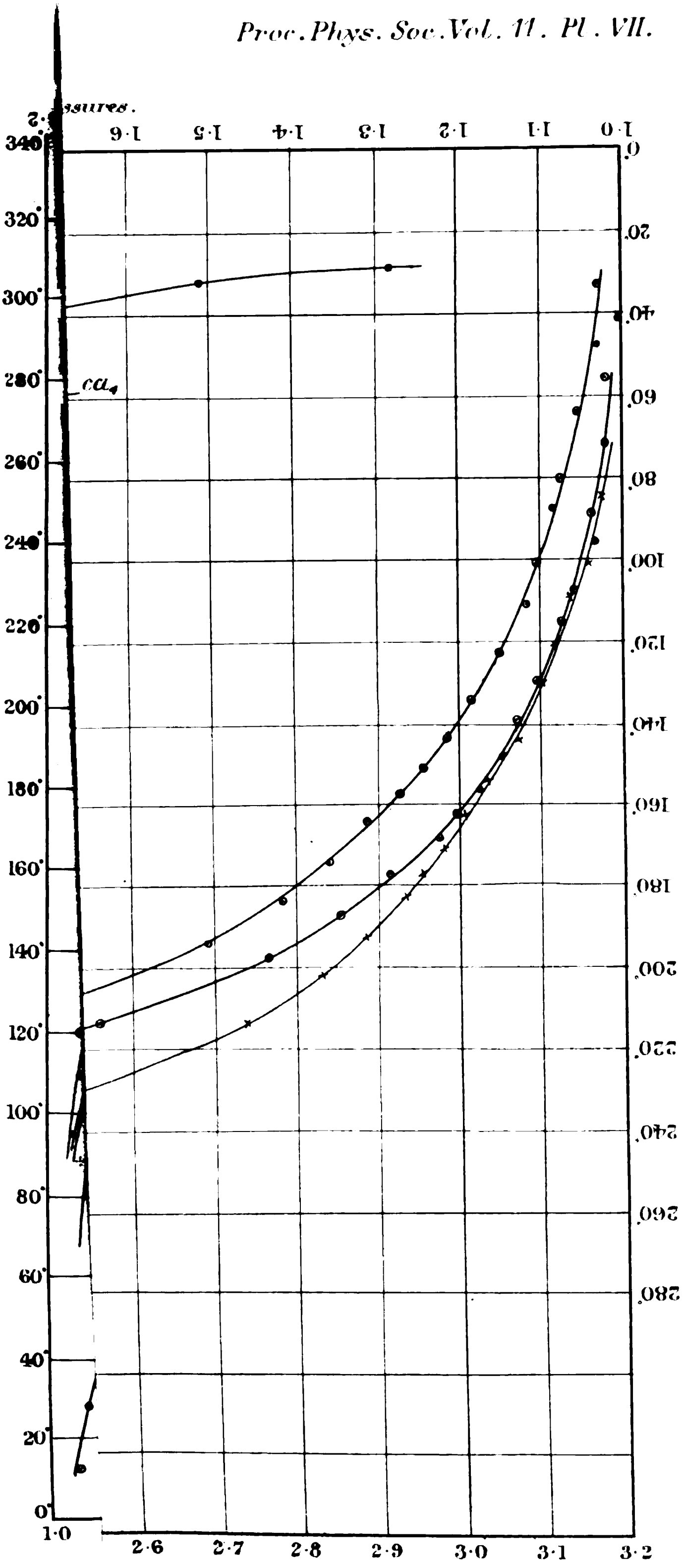
‡ Ibid. clxxx. p. 137.

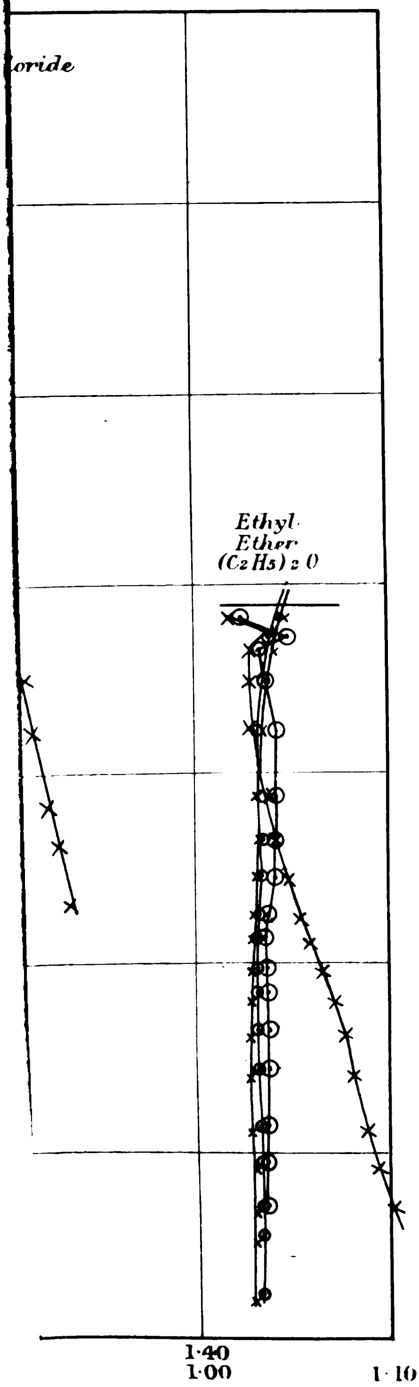
§ Ibid. clxxviii. A, p. 57.

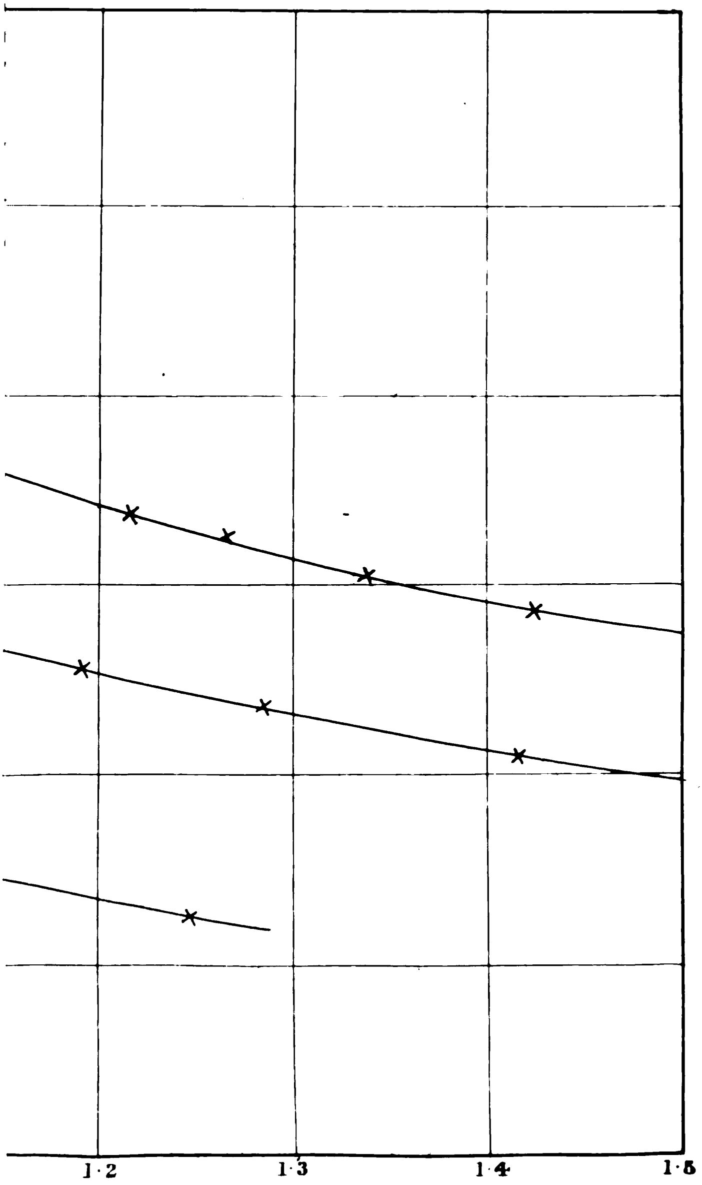
‖ Trans. Chem. Soc. xlix. p. 790.

The experimental data and the ratios calculated from them are given in the following tables.

I. Critical constants—temperature, pressure, volume.
II. Corresponding pressures.
III. Corresponding temperatures.
IV. Boiling-points on absolute scale at corresponding pressures.
V. Vapour-pressures at corresponding temperatures.
VI. Molecular volumes of liquids at corresponding pressures.
VII. Molecular volumes of liquids at corresponding temperatures.
VIII. Molecular volumes of saturated vapour at corresponding pressures.
IX. Molecular volumes of saturated vapour at corresponding temperatures.
X. Ratios of absolute temperatures to those of fluorbenzene at corresponding pressures.
XI. Ratios of vapour-pressures at corresponding temperatures.
XII. Ratios of molecular volumes of liquid at corresponding pressures.
XIII. Ratios of molecular volumes of liquid at corresponding temperatures.
XIV. Ratios of molecular volumes of saturated vapour at corresponding pressures.


340°
320°
300°
280°
260°
240°
220°
200°
180°
160°
140°
120°
100°
80°
60°
40°
20°
0°
1·0
2·6
2·7
2·8
2·9
3·0
3·1
3·2
ca₄
1·6
1·5
1·4
1·3
1·2
1·1
1·0
0
20°
40°
60°
80°
100°
120°
140°
160°
180°
200°
220°
240°
260°
280°


oride
Ethyl
Ether
(C2 H5) 2 0
1·40
1·00
1·10


1·2
1·3
1·4
1·5

XV. Ratios of molecular volumes of saturated vapour at corresponding temperatures.

In calculating the molecular volumes the following molecular weights have been employed:—

C_6H_5F	... 95·8	C_6H_5I	... 203·4	$SnCl_4$	......259·3	C_2H_5OH	...45·90
C_6H_5Cl	... 112·2	C_6H_6	... 77·84	$(C_2H_5)_2O$	... 73·84	C_3H_7OH	...59·87
C_6H_5Br	... 156·6	CCl_4	... 153·45	CH_3OH	... 31·93	CH_3COOH	59·86

TABLE I.—Critical Constants.

Substance.	Formula.	Temperature.		Pressure, in millim.	Volume, in c.c.	
		Centigrade.	Absolute		of a gram.	Molecular.
Fluorbenzene.........	C_6H_5F.	286·55	559·55	33912	2·43	233
Chlorobenzene	C_6H_5Cl.	(360)	(633)	(33912)	(2·34)	(262)
Bromobenzene	C_6H_5Br.	(397)	(670)	(33912)	(1·76)	(275)
Iodobenzene	C_6H_5I.	(448)	(721)	(33912)	(1·47)	(298)
Benzene	C_6H_6.	288·5	561·5	36395	2·82	219
Carbon tetrachloride	CCl_4.	283·15	556·15	34180		
Stannic Chloride ...	$SnCl_4$.	318·7	591·7	28080		
Ether	$(C_2H_5)_2O$.	194·4	467·4	27060		
Methyl Alcohol......	CH_3OH.	240·0	513·0	59760		
Ethyl Alcohol	C_2H_5OH.	243·1	516·1	47850		
Propyl Alcohol	C_3H_7OH.	263·7	536·7	38120		
Acetic Acid	CH_3COOH.	321·6	594·6	43400	2·46	147

The brackets indicate calculated values. In the case of chlorobenzene, bromobenzene, and iodobenzene the critical temperatures and volumes given depend on the assumption that the critical pressures are equal. The critical constants of chlorobenzene have been observed, but the determinations could not be made with the same degree of accuracy as those of fluorbenzene, and the calculated values have therefore been adopted in this paper. The observed values are:—temperature, 359·2 to 359·45; pressure, 33926 to 33998 millim.; molecular volume, 262 to 275 cubic centim. The temperatures given in the original paper (Trans. Chem. Soc. lv. p. 518) are 360·55–360·8; but the boiling-point of mercury, the vapour of which was employed as a jacket, has since been shown to be lower than was at that time adopted.

TABLE II.

Corresponding Pressures.

Halogen Derivatives of Benzene.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	21·46	20·16	16·56	15·96	35·24	28·22	22·48	25·60
50	53·66	50·40	41·40	39·90	88·11	70·55	56·21	63·99
100	107·3	100·8	82·24	79·80	176·2	141·1	112·4	128·0
200	214·6	201·6	165·6	159·6	352·4	282·2	224·8	256·0
400	429·3	403·2	331·2	319·2	704·9	564·4	449·7	511·9
760	815·6	766·0	629·4	606·5	1339	1072	854·3	972·6
1000	1073·0	1008	822·4	798·0	1762	1411	1124	1280
1500	1610	1512	1242	1197	2643	2116	1686	1920
2000	2146	2016	1656	1596	3524	2822	2248	2560
3000	3220	3023	2484	2394	5287	4233	3372	3839
4000	4293	4032	3312	3192	7049	5644	4497	5119
5000	5366	5040	4140	3990	8811	7055	5621	6399
6000	6439	6048	4920	4787	10573	8467	6745	7679
7000	7513	7055	5796	5586	12337	9876	7869	8959
8000	8586	8063	6624	6384	14096	11288	8993	10238
10000	10732	10079	8224	7980	17624	14109	11241	12798
12000	12879	12095	9936	9576	21145	16932	13490	15357
15000	16098	15118	12419	11970	26430	21164	16862	19197
20000	21464	20156	16562	15962	35245	28223	22480	25596
25000	26831	25200	20701	19948	44055	35278	28100	31995
28000	30047	28223	23185	22341	49340	39509	31477	35834
30000	32197	30234	24843	23939	52869	42325	33721	38394
32000	34340	32255	26497	25533	56390	45154	35975	40953
33912	36395	34182	28080	27060	59760	47850	38120	43400

TABLE III.

Corresponding Temperatures (absolute).

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	272·25	308·0	326·0	350·8	273·2	270·6	287·9	227·4	249·6	251·1	261·15	289·3
50	289·3	327·3	346·4	372·8	290·3	287·55	305·9	241·65	265·25	266·85	277·5	307·4
100	303·9	343·8	363·9	391·6	304·95	302·05	321·35	253·85	278·6	280·3	291·5	322·95
200	320·25	362·3	383·45	412·65	321·35	318·3	338·65	267·5	293·6	295·4	307·15	340·3
400	338·75	383·2	405·6	436·5	339·95	336·7	358·2	282·95	310·55	312·45	324·9	359·95
760	358·1	405·1	428·8	461·4	359·35	355·9	378·65	299·15	328·3	330·3	343·45	380·5
1000	367·3	415·5	439·8	473·3	368·6	365·05	388·4	306·8	336·75	338·75	352·3	390·3
1500	382·0	432·15	457·4	492·2	383·35	379·65	403·95	319·1	350·2	352·35	366·4	405·9
2000	393·25	444·85	470·85	506·7	394·6	390·85	415·85	328·5	360·55	362·7	377·2	417·9
3000	410·4	464·25	491·4	528·8	411·85	407·9	434·0	342·8	376·25	378·55	393·65	436·1
4000	423·8	479·4	507·45	546·1	425·3	421·2	448·15	354·0	388·55	390·9	406·5	450·35
5000	434·85	491·95	520·7	......	436·35	432·2	459·85	363·25	398·7	401·1	417·1	462·1
6000	444·25	502·55	531·95	......	445·8	441·55	469·8	371·1	407·3	409·75	426·1	472·1
7000	452·8	512·25	542·2	......	454·4	450·05	478·8	378·25	415·15	417·65	434·3	481·15
8000	460·4	520·85	551·3	......	462·0	457·6	486·85	384·6	422·1	424·65	441·6	489·25
10000	473·6	535·75	......	......	475·25	470·7	500·8	395·6	434·2	436·8	454·25	503·25
12000	484·95	548·6	......	......	486·65	482·0	512·8	405·1	444·6	447·3	465·15	515·3
15000	499·7	565·3	......	......	501·45	496·65	528·4	417·4	458·15	460·9	479·3	531·0
20000	519·7	587·9	......	......	521·5	516·55	549·55	434·1	476·45	479·35	498·45	552·25
25000	536·0	606·35	......	......	537·85	532·75	566·8	447·7	491·4	494·35	514·1	569·55
28000	544·5	615·95	......	......	546·4	541·2	575·8	454·85	499·2	502·2	522·25	578·6
30000	550·0	622·2	......	......	551·9	546·65	581·6	459·4	504·25	507·3	527·55	584·45
32000	555·0	627·85	......	......	556·95	551·6	586·9	463·6	508·85	511·9	532·3	589·75
33912	559·55	(633·0)	(670·0)	(721·0)	561·5	556·15	591·7	467·4	513·0	516·1	536·7	594·6

TABLE IV.

Boiling-points on Absolute Scale at corresponding Pressures.

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	272·25	308·05	326·3	351·25	269·25	264·05	290·75	......	275·7	285·7	299·4	307·05
50	289·3	327·1	346·3	372·75	286·25	281·05	308·65	......	291·65	301·2	314·55	325·7
100	303·9	343·4	363·6	391·3	300·9	295·5	323·7	......	304·85	314·05	327·45	341·5
200	320·25	361·95	383·05	412·2	317·15	311·9	341·0	270·0	319·55	328·3	341·6	359·2
400	338·75	382·8	405·3	436·1	336·3	330·5	360·3	285·0	336·0	344·0	357·25	378·9
760	358·1	404·9	429·0	461·55	355·65	350·0	380·75	301·2	353·05	360·3	373·55	399·3
1000	367·3	415·4	440·05	473·5	364·95	359·25	390·05	308·7	361·0	367·75	381·25	408·9
1500	382·0	432·2	457·8	492·45	379·85	374·15	405·6	320·9	373·3	379·6	393·2	424·1
2000	393·25	445·0	471·3	507·0	391·3	385·7	417·25	330·2	381·6	388·7	402·25	435·8
3000	410·4	464·6	492·0	529·0	408·9	403·25	435·3	344·5	396·75	402·25	415·2	453·3
4000	423·8	479·9	507·95	545·65	422·5	416·8	449·2	355·55	407·6	412·7	425·95	466·6
5000	434·85	492·45	521·05	......	433·75	428·05	460·8	364·5	416·5	421·2	435·7	477·5
6000	444·25	503·0	532·1	......	443·5	437·75	470·3	372·4	424·0	428·5	443·25	486·7
7000	452·8	512·5	542·05	......	452·15	446·3	479·6	379·2	430·7	435·0	449·95	494·8
8000	460·4	520·95	550·9	......	459·8	454·15	487·35	385·35	436·7	440·95	456·0	502·05
10000	473·6	535·5	......	......	473·3	467·8	500·75	396·0	446·75	450·95	466·4	514·4
12000	484·95	548·2	......	......	485·0	479·5	513·05	405·25	455·5	459·6	475·6	525·05
15000	499·7	564·6	......	......	500·05	494·35	528·6	417·05	466·75	470·65	487·4	538·5
20000	519·7	587·25	......	......	520·5	515·0	549·5	433·6	481·25	485·75	503·6	557·0
25000	536·0	605·7	......	......	537·1	531·95	566·7	447·45	494·7	498·15	517·2	572·35
28000	544·5	615·25	......	......	546·05	540·85	575·85	454·7	501·45	504·5	524·4	580·5
30000	550·0	621·5	......	......	551·6	546·25	581·55	459·2	505·55	508·65	529·0	585·3
32000	555·0	627·2	......	......	556·75	551·5	586·75	463·45	509·35	512·55	533·15	590·2
33912	559·55	632·5	(670·0)	(721·0)	561·5	556·15	591·7	467·4	513·0	516·1	536·7	594·6

TABLE V.

Vapour-pressures at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
272·25	20	19·95	19·75	19·70	26·76	29·21	14·13	......	......	......	......	9·53
289·3	50	50·40	50·10	50·10	65·46	69·50	36·35	......	17·95	......	4·95	24·77
303·9	100	101·7	101·2	101·0	128·8	134·6	74·13	......	41·69	20·13	13·34	56·23
320·25	200	203·0	202·1	202·5	252·9	259·4	151·5	140·9	98·63	50·64	36·48	122·3
338·75	400	405·5	403·2	402·7	495·5	501·2	308·7	291·1	231·7	129·1	98·74	263·0
358·1	760	763·8	758·6	758·6	912·0	916·2	591·6	562·3	515·2	310·5	245·5	538·3
367·3	1000	1001	994·3	997·7	1193	1189	784·3	746·4	726·9	452·9	364·8	739·6
382·0	1500	1498	1491	1495	1766	1740	1191	1130	1211	790·7	651·6	1176
393·25	2000	1995	1982	1989	2325	2272	1598	1515	......	1175	973·9	1635
410·4	3000	2979	2965	2992	3432	3350	2415	2283	2897	2042	1718	2576
423·8	4000	3967	3967	4027	4545	4406	3240	3076	4198	3027	2550	3589
434·85	5000	4949	4972	......	5636	5451	4064	3868	5559	4102	3455	4656
444·25	6000	5943	5977	......	6722	6464	4881	4651	6998	5218	4411	5735
452·8	7000	6982	7015	......	7834	7516	5715	5476	8521	6449	5445	6894
460·4	8000	7998	8054	......	8913	8531	6569	6281	10070	7700	6479	8063
473·6	10000	10025	......	......	11079	10556	8232	7907	13351	10292	8580	10471
484·95	12000	12050	......	......	13228	12560	9897	9539	16788	13017	10952	13002
499·7	15000	15136	......	......	16425	15613	12402	12023	22284	17398	14504	16943
519·7	20000	20137	......	......	21752	20583	16558	16088	31659	25061	20559	23796
536·0	25000	25206	......	......	27071	25468	20725	20045	41543	33037	26761	30726
544·5	28000	28184	......	......	30200	28347	23174	22413	47479	37844	30374	34954
550·0	30000	30234	......	......	32322	30374	24860	23988	51761	41400	32584	37932
555·0	32000	32211	......	......	34435	32285	26516	25615	55847	44668	35481	40691
559·55	33912	(33912)	(33912)	(33912)	36395	34180	28080	27060	59760	47850	58120	43400

TABLE VI.

Molecular Volumes of Liquid at corresponding Pressures.

Pressure. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	91·47	102·89	107·99	116·79	86·10	......	116·17	......	......	57·69	74·99	57·85
50	93·30	104·88	110·04	119·03	87·87	94·91	118·66	......	40·27	58·66	76·19	59·00
100	94·92	106·63	111·89	121·09	89·41	96·58	120·78	......	40·77	59·51	77·29	60·06
200	96·80	108·75	114·10	123·52	91·23	98·50	123·32	......	41·56	60·51	78·56	61·33
400	99·05	111·32	116·81	126·48	93·51	100·86	126·51	102·21	42·43	61·72	80·06	62·85
760	101·59	114·26	119·94	129·86	95·91	103·52	129·80	104·94	43·42	63·11	81·81	64·59
1000	102·90	115·75	121·47	131·56	97·12	104·87	131·52	106·31	43·92	63·80	82·70	65·46
1500	105·10	118·23	124·11	134·40	99·20	107·19	134·47	108·60	44·76	64·97	84·20	66·91
2000	107·00	120·24	126·24	136·72	100·96	109·10	136·88	110·46	45·45	65·97	85·47	68·09
3000	110·03	123·69	129·86	140·63	103·89	112·23	140·88	113·49	46·60	67·65	87·65	70·01
4000	112·64	126·65	132·90	143·73	106·35	114·86	144·09	116·08	47·56	69·10	89·50	71·61
5000	114·98	129·27	135·68	......	108·50	117·16	147·10	118·44	48·42	70·48	91·24	73·04
6000	117·06	131·72	138·23	......	110·52	119·29	149·82	120·70	49·23	71·67	92·87	74·38
7000	119·14	134·08	140·75	......	112·48	121·32	152·62	122·91	50·07	72·89	94·45	75·69
8000	121·19	136·33	143·18	......	114·36	123·38	155·24	125·12	50·88	73·97	96·03	76·94
10000	125·04	140·72	......	......	117·95	127·39	160·27	129·33	52·36	76·15	99·01	79·37
12000	128·80	145·10	......	......	121·55	131·39	165·56	133·36	53·87	78·33	101·99	81·79
15000	134·64	......	......	......	127·12	137·34	173·37	139·23	56·17	81·63	106·57	85·30
20000	145·08	......	......	......	137·20	148·19	187·29	149·82	60·54	88·22	114·89	91·85
25000	158·40	179·2	......	......	149·80	161·89	......	164·15	66·10	96·34	125·67	100·1
28000	169·35	191·5	......	......	160·20	173·55	......	176·18	70·92	102·59	134·41	107·4
30000	179·40	202·2	......	......	169·50	183·37	......	186·52	75·04	109·10	142·52	113·4
32000	193·0	219·5	......	......	182·3	196·26	......	201·80	81·49	118·01	153·99	121·8
33912	233·0	......	......	......	219·0	......	......	......	......	......	......	147·0

Approximate.

TABLE VII.

Molecular Volumes of Liquid at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
272·25	91·47	102·88	107·95	116·73	86·50	......	115·78	......	......	......	......	56·85
289·3	93·30	104·89	110·04	119·03	88·28	95·63	118·24	......	......	......	73·38	57·87
303·9	94·92	106·69	111·93	121·12	89·86	97·33	120·45	......	......	57·37	74·39	58·83
320·25	96·80	108·80	114·13	123·57	91·73	99·31	122·98	......	40·36	58·29	75·59	59·99
338·75	99·05	111·37	116·86	126·50	93·95	101·69	125·99	101·86	41·12	59·40	77·09	61·38
358·1	101·59	114·17	119·93	129·85	96·38	104·36	129·47	104·60	42·01	60·66	78·71	62·97
367·3	102·90	115·77	121·44	131·54	97·62	105·76	131·18	105·97	42·47	61·31	79·56	63·81
382·0	105·10	118·21	124·06	134·36	99·73	108·08	134·13	108·26	43·25	62·42	81·02	65·19
393·25	107·00	120·22	126·17	136·67	101·48	109·99	136·55	110·10	43·89	63·33	82·22	66·30
410·4	110·03	123·63	129·76	140·59	104·41	113·11	140·57	113·10	44·97	64·86	84·29	68·12
423·3	112·64	126·56	132·79	144·05	106·90	115·73	143·83	115·72	45·92	66·23	86·11	69·67
434·85	114·98	129·17	135·60	......	109·04	118·03	146·87	118·11	46·77	67·50	87·80	71·05
444·25	117·06	131·60	138·18	......	111·05	120·15	149·67	120·32	47·54	68·68	89·42	72·32
452·8	119·14	134·05	140·77	......	113·05	122·28	152·39	122·61	48·29	69·86	91·00	73·56
460·4	121·19	136·33	143·30	......	114·90	124·31	155·09	124·86	49·06	71·03	92·50	74·79
473·6	125·04	140·79	......	......	118·51	128·32	160·32	129·18	50·53	73·21	95·61	77·16
484·95	128·80	145·29	......	......	122·07	132·28	165·43	133·24	52·03	75·37	98·67	79·57
499·7	134·64	......	......	......	127·68	138·29	173·24	139·02	54·36	78·70	103·37	83·31
519·7	145·08	......	......	......	137·81	149·16	187·34	150·26	58·72	85·01	111·87	90·00
536·0	158·40	180·0 } Approximate.	......	......	150·60	162·81	......	164·37	64·31	93·52	122·55	98·25
544·5	169·35	192·4	......	......	160·70	174·09	......	176·48	69·13	99·92	131·41	105·45
550·0	179·40	204·0	......	......	170·00	184·06	......	187·11	73·63	106·58	139·50	112·15
555·0	193·0	222·4	......	......	183·00	196·57	......	202·32	80·43	116·13	150·87	120·8
559·55	233·0	......	......	......	219·0	......	......	......	......	......	......	147

TABLE VIII.

Molecular Volumes of Saturated Vapour at corresponding Pressures.

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	......	......	......	......	......	......	......	......	......	......	......	391000
50	......	......	......	......	......	......	......	......	......	......	......	173000
100	......	......	......	......	......	......	......	......	105000	138000	......	93700
200	......	......	......	......	......	......	......	......	54600	71550	......	50750
400	......	......	......	......	......	......	......	53900	28210	37150	48150	27410
760	......	......	......	......	......	......	36200	29500	15290	20210	25950	15400
1000	22000	24650	25750	27950	20400	21150	28150	22800	11740	15550	19880	12010
1500	15000	16900	17600	19200	13900	14550	19100	15550	7945	10500	13410	8275
2000	11400	12900	13400	14600	10630	11120	14540	11790	5997	7960	10120	6340
3000	7680	8760	9130	9900	7235	7550	9840	7970	4032	5354	6760	4341
4000	5785	6615	6910	7465	5500	5740	7445	6005	3022	4024	5056	3306
5000	4634	5308	5544	......	4421	4618	5943	4811	2410	3218	4041	2673
6000	3857	4411	4624	......	3694	3864	4995	4005	1995	2872	3355	2244
7000	3298	3771	3954	......	3167	3292	4232	3420	1698	2275	2861	1927
8000	2871	3281	3445	......	2759	2864	3398	2976	1474	1974	2486	1688
10000	2265	2576	......	......	2175	2257	2948	2358	1158	1558	1961	1349
12000	1862	2106	......	......	1778	1851	2388	1936	943	1256	1596	1116
15000	1447	......	......	......	1371	1440	1841	1508	728	962	1229	879
20000	1009	......	......	......	953	1012	1274	1052	499	651	837	628
25000	733	......	......	......	689	733	......	759	352	468	594	457
28000	601	......	......	......	563	597	......	622	283	386	489	374
30000	516	......	......	......	489	519	......	542	245	332	426	325
32000	440	......	......	......	......	447	......	454	208	275	368	275
33912	233	......	......	......	219	......	......	......	......	......	......	147

TABLE IX.

Molecular Volumes of Saturated Vapour at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
272·25	......	......	......	......	......	......	......	......	......	......	......	......
289·3	......	......	......	......	......	......	......	......	......	......	......	384500
303·9	......	......	......	......	......	......	......	......	......	......	......	193500
320·25	......	......	......	......	......	......	......	......	182000	......	......	99700
338·75	......	......	......	......	......	......	......	58600	81300	150500	......	49500
358·1	......	......	......	......	......	......	38350	31650	38000	65300	......	26150
367·3	22000	24650	25850	28000	18550	18250	29400	24250	27400	45700	58950	19680
382·0	15000	16920	17770	19280	12780	12760	19910	16400	16850	27050	33800	12940
393·25	11400	12930	13540	14710	9865	9865	15050	12400	11950	18600	22850	9625
410·4	7680	8830	9225	9930	6800	6870	10100	8320	7260	10850	13170	6280
423·8	5785	6653	6958	7413	5164	5266	7595	6237	5065	7430	8915	4613
434·85	4634	5353	5576	......	4198	4281	6060	4972	3828	5521	6592	3622
444·25	3857	4457	4645	......	3544	3597	5035	4116	3041	4345	5158	2968
452·8	3298	3780	3954	......	3023	3083	4285	3491	2489	3532	4173	2495
460·4	2871	3281	3414	......	2649	2692	3733	3027	2094	2951	3491	2138
473·6	2265	2567	......	......	2099	2150	2944	2374	1560	2178	2597	1652
484·95	1862	2077	......	......	1734	1776	2404	1939	1219	1690	2014	1326
499·7	1447	......	......	......	1340	1388	1845	1505	890	1218	1471	1005
519·7	1609	......	......	......	934	986	1274	1040	578	770	952	687
536·0	733	......	......	......	677	719	......	753	389	516	641	487
544·5	601	......	......	......	559	592	......	619	305	415	518	393
550·0	516	......	......	......	485	514	......	538	257	349	446	335
555·0	440	......	......	......	......	446	......	451	213	285	381	281
559·55	233	......	......	......	219	......	......	......	......	......	......	147

TABLE X.

Ratios of Absolute Temperatures (Boiling-Points) to those of Fluorbenzene at corresponding Pressures.

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	1·0000	1·1315	1·1985	1·2902	·9890	·9699	1·0679	……	1·0127	1·0494	1·0097	1·1278
50	„	1·1307	1·1970	1·2885	·9895	·9715	1·0669	……	1·0081	1·0411	1·0873	1·1258
100	„	1·1300	1·1964	1·2876	·9901	·9724	1 0651	……	1·0031	1·0334	1·0775	1·1237
200	„	1·1302	1·1961	1·2871	·9903	·9739	1·0648	·8431	·9978	1·0251	1·0667	1·1216
400	„	1·1300	1·1965	1·2874	·9928	·9756	1·0636	·8413	·9919	1·0155	1·0546	1·1185
760	„	1·1307	1·1980	1·2889	·9932	·9774	1·0633	·8411	·9859	1·0061	1·0432	1·1151
1000	„	1·1310	1·1981	1·2891	·9936	·9781	1·0617	·8405	·9829	1·0012	1·0380	1·1133
1500	„	1·1314	1·1985	1·2891	·9944	·9795	1 0618	·8401	·9772	·9937	1·0293	1·1102
2000	„	1·1316	1·1985	1·2893	·9950	9808	1 0613	·8397	·9704	·9884	1·0229	1·1082
3000	„	1·1321	1·1988	1·2890	·9963	·9826	1·0607	·8394	·9667	·9801	1·0117	1·1045
4000	„	1·1324	1·1986	1·2875	·9969	·9835	1·0599	·8390	·9618	·9738	1·0051	1·1010
5000	„	1·1325	1·1982	……	·9975	·9844	1·0597	·8382	·9578	·9686	1·0020	1·0981
6000	„	1·1322	1·1977	……	·9983	·9854	1·0586	·8383	·9544	·9645	·9977	1·0955
7000	„	1·1318	1·1971	……	·9986	·9856	1·0592	·8375	·9512	·9607	·9937	1·0927
8000	„	1·1315	1·1965	……	·9987	·9864	1·0585	·8370	·9485	·9577	·9904	1·0905
10000	„	1·1307	……	……	·9994	·9878	1·0573	·8362	·9433	·9522	·9848	1·0862
12000	„	1·1304	……	……	1·0001	·9888	1·0579	·8356	·9393	·9477	·9807	1·0827
15000	„	1·1299	……	……	1·0007	·9893	1·0578	·8346	·9340	·9419	·9754	1·0777
20000	„	1·1300	……	……	1·0015	·9910	1 0574	·8343	·9260	·9847	·9690	1·0718
25000	„	1·1301	……	……	1·0021	·9925	1·0573	·8348	·9220	·9294	·9649	1·0678
28000	„	1·1299	……	……	1·0028	·9933	1·0576	·8351	·9209	·9265	·9631	1·0661
30000	„	1·1300	……	……	1·0029	·9932	1·0574	·8349	·9192	·9248	·9618	1·0642
32000	„	1·1301	……	……	1·0032	·9937	1·0572	·8350	·9178	·9235	·9606	1·0634
33912	„	……	……	……	1·0035	·9939	1·0575	·8353	·9168	·9223	·9592	1·0626
Mean Ratios.												
		1·1309	1·1976	1·2885								
Ratio of Highest to Lowest Ratio.												
		1·002	1·002	1·002	1·014	1·025	1·010	1·011	1·105	1·138	1·146	1·061

TABLE XI.

Ratios of Vapour-Pressures to those of Fluorbenzene at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl,	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
272·25	1·0000	·998	·987	·985	1·338	1·460	·706	......	......	......	......	·476
289·3	„	1·008	1·012	1·002	1·309	1·390	·727	......	·359	......	·090	·495
303·9	„	1·016	1·012	1·010	1·288	1·346	·741	......	·417	·201	·133	·562
320·25	„	1·015	1·010	1·013	1·265	1·297	·758	·705	·493	·253	·182	·612
338·75	„	1·014	1·008	1·007	1·239	1·253	·772	·728	·579	·323	·247	·658
358·1	„	1·005	·998	·998	1·200	1·206	·778	·740	·678	·409	·323	·708
367·3	„	1·001	·994	·998	1·193	1·159	·784	·746	·727	·453	·365	·740
382·0	„	·999	·994	·996	1·177	1·160	·794	·753	·807	·527	·434	·784
393·25	„	·998	·991	·994	1·163	1·136	·799	·758	......	·587	·487	·817
410·4	„	·993	·988	·997	1·144	1·117	·805	·761	·966	·681	·573	·859
423·8	„	·992	·992	1·007	1·136	1·101	·810	·769	1·049	·757	·637	·897
434·85	„	·990	·994	......	1·127	1·090	·813	·774	1·112	·820	·691	·931
444·25	„	·991	·996	......	1·120	1·077	·814	·775	1·167	·870	·735	·956
452·8	„	·997	1·002	......	1·119	1·074	·816	·782	1·217	·921	·778	·985
460·4	„	1·000	1·007	......	1·114	1·066	·821	·785	1·259	·962	·810	1·008
473·6	„	1·002	......	......	1·108	1·056	·823	·791	1·335	1·029	·858	1·047
481·95	„	1·004	......	......	1·102	1·047	·825	·795	1·399	1·085	·913	1·083
499·7	„	1·009	......	......	1·095	1·041	·827	·801	1·486	1·160	·967	1·130
519·7	„	1·007	......	......	1·088	1·029	·828	·804	1·583	1·253	1·028	1·190
536·0	„	1·008	......	......	1·083	1·019	·829	·802	1·662	1·322	1·071	1·229
544·5	„	1·006	......	......	1·078	1·012	·828	·800	1·696	1·351	1·085	1·248
550·0	„	1·008	......	......	1·077	1·013	·829	·800	1·725	1·380	1·086	1·264
555·0	„	1·007	......	......	1·076	1·009	.829	·801	1·745	1·396	1·109	1·272
559·55	„	......	......	......	1·073	1·008	·828	·798	1·762	1·411	1·124	1·280
Ratio of Highest to Lowest Ratio.		1·026	1·025	1·028	1·247	1·448	1·174	1·140	4·91	7·02	11·35	2·68

TABLE XII.

Ratios of Molecular Volumes of Liquid to those of Fluorbenzene at corresponding Pressures.

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	1·0000	1·1248	1·1806	1·2768	·9413	......	1·2700	.. ...	......	·6307	·8198	·6324
50	,,	1·1241	1·1794	1·2758	·9418	1·0173	1·2718	......	·4317	·6287	·8166	·6324
100	,,	1·1234	1·1788	1·2757	·9420	1·0175	1·2724	......	·4295	·6269	·8143	·6327
200	,,	1·1234	1·1787	1·2760	·9425	1·0176	1·2740	......	·4293	·6251	·8115	·6336
400	,,	1·1239	1·1793	1·2769	·9441	1·0183	1·2773	1·0319	·4294	·6232	·8083	·6345
760	,,	1·1247	1·1806	1·2783	·9441	1·0190	1·2777	1·0330	·4274	·6213	·8053	·6358
1000	,,	1·1249	1·1805	1·2785	·9438	1·0191	1·2781	1·0332	·4268	·6200	·8037	·6362
1500	,,	1·1249	1·1809	1·2788	·9439	1·0199	1·2795	1·0333	·4259	·6182	·8012	·6366
2000	,,	1·1238	1·1798	1·2778	·9436	1·0197	1·2793	1·0324	·4247	·6166	·7987	·6364
3000	,,	1·1241	1·1802	1·2781	·9442	1·0200	1·2804	1·0315	·4235	·6148	·7966	·6363
4000	,,	1·1244	1·1799	1·2760	·9442	1·0197	1·2792	1·0305	·4222	·6137	·7953	·6357
5000	,,	1·1243	1·1800	......	·9436	1·0190	1·2794	1·0301	·4211	·6130	·7936	·6359
6000	,,	1·1252	1·1808	......	·9441	1·0191	1·2799	1·0311	·4210	·6123	·7934	·6354
7000	,,	1·1254	1·1814	......	·9441	1·0183	1·2810	1·0317	·4202	·6118	·7927	·6353
8000	,,	1·1249	1·1814	......	·9436	1·0180	1·2810	1·0324	·4198	·6103	·7924	·6349
10000	,,	1·1254	......	......	·9433	1·0188	1·2818	1·0343	·4188	·6090	·7918	·6348
12000	,,	1·1265	......	......	·9437	1·0201	1·2854	1·0354	·4182	·6081	·7918	·6350
15000	,,	......	......	......	·9442	1·0200	1 2877	1·0341	·4172	·6063	·7915	·6335
20000	,,	......	......	......	·9457	1·0214	1·2910	1·0327	·4173	·6081	·7919	·6331
25000	,,	1·1310 Approximate.	......	......	·9457	1·0220	......	1·0364	·4173	·6082	·7933	·6319
28000	,,	1·1310	......	......	·9460	1·0248	......	1·0403	·4188	·6058	·7937	·6342
30000	,,	1·1270	......	......	·9448	1·0222	......	1·0397	·4183	·6082	·7944	·6321
32000	,,	1·1350	......	......	·9446	1·0169	......	1·0456	·4222	·6114	·7978	·6311
33912	,,	......	......	......	·9400	......	......	......	......	......	. ....	·6310
Mean Ratios.		*1·1246	1·1802	1·2772	·9439	1·0195	1·2793	1·0342	·4228	·6153	·7995	·6342
Ratio of Highest to Lowest Ratio.		*1·003	1·002	1·002	1·006	1·008	1·017	1·015	1·035	1·041	1·036	1·009

* Omitting approximate values.

TABLE XIII.

Ratios of Molecular Volumes of Liquid to those of Fluorbenzene at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
272·25	1·0000	1·1247	1·1802	1·2761	·9457	......	1·2657	......	......	......	......	·6215
289·3	,,	1·1242	1·1794	1·2758	·9462	1·0250	1·2673	......	......	......	·7865	·6203
303·9	,,	1·1240	1·1792	1·2760	·9467	1·0254	1·2689	......	......	·6044	·7837	·6198
320·25	,,	1·1239	1·1790	1·2765	·9476	1·0260	1·2705	......	·4169	·6022	·7809	·6197
338·75	,,	1·1244	1·1798	1·2772	·9485	1·0267	1·2720	1·0284	·4152	·5997	·7783	·6197
358·1	,,	1·1238	1·1805	1·2782	·9487	1·0273	1·2744	1·0297	·4136	·5971	·7748	·6198
367·3	,,	1·1250	1·1802	1·2783	·9487	1·0278	1·2748	1·0299	·4127	·5958	·7732	·6201
382·0	,,	1·1248	1·1804	1·2784	·9489	1·0283	1·2763	1·0301	·4115	·5939	·7709	·6203
393·25	,,	1·1236	1·1792	1·2773	·9484	1·0280	1·2761	1·0290	·4102	·5919	·7684	·6196
410·4	,,	1·1236	1·1793	1·2777	·9489	1·0280	1·2775	1·0279	·4087	·5895	·7661	·6191
423·8	,,	1·1236	1·1789	1·2789	·9491	1·0275	1·2769	1·0273	·4076	·5880	·7664	·6185
434·85	,,	1·1234	1·1793	......	·9484	1·0266	1·2773	1·0272	·4068	·5871	·7636	·6179
444·25	,,	1·1242	1·1804	......	·9487	1·0264	1·2786	1·0279	·4061	·5867	·7639	·6178
452·8	,,	1·1252	1·1815	......	·9489	1·0264	1·2791	1·0291	·4054	·5864	·7638	·6174
460·4	,,	1·1249	1·1824	......	·9481	1·0257	1·2797	1·0302	·4048	·5861	·7633	·6171
473·6	,,	1·1260	......	......	·9478	1·0262	1·2822	1·0331	·4041	·5855	·7646	·6171
484·95	,,	1·1280	......	......	·9477	1·0270	1·2844	1·0345	·4040	·5851	·7660	·6178
499·7	,,	......	......	......	·9483	1·0271	1·2867	1·0326	·4038	·5845	·7677	·6188
519·7	,,	......	......	......	·9499	1·0281	1·2913	1·0357	·4047	·5859	·7711	·6204
536·0	,,	1·1360 Approximate.	......	......	·9508	1·0279	......	1·0377	·4060	·5904	·7737	·6203
544·5	,,	1·1360	......	......	·9489	1·0280	......	1·0421	·4082	·5900	·7760	·6227
550·0	,,	1·1370	......	......	·9476	1·0260	......	1·0430	·4104	·5941	·7776	·6251
555·0	,,	1·1520	......	......	·9482	1·0188	......	1·0483	·4167	·6017	·7817	·6259
559·55	,,	......	......	......	·9400	......	......	......	......	......	......	·6310
Mean Ratios.												
		*1·1245	1·1800	1·2773	·9479	1·0266	1·277	1·033	·409	·592	·772	·623
Ratio of Highest to Lowest Ratio.												
		*1·004	1·003	1·002	1·011	1·009	1·020	1·021	1·032	1·030	1·034	1·023

* Omitting approximate values.

TABLE XIV.

Ratios of Molecular Volumes of Saturated Vapour to those of Fluorbenzene at corresponding Pressures.

Pressure C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
1000	1·000	1·120	1·168	1·269	·925	·960	1·278	1·036	·533	·706	·903	·545
1500	„	1·128	1·175	1·279	·930	·970	1·276	1·037	·530	·700	·895	·552
2000	„	1·133	1·179	1·283	·933	·976	1·277	1·036	·527	·701	·888	·557
3000	„	1·141	1·189	1·289	·942	·983	1·281	1·038	·525	·697	·880	·565
4000	„	1·143	1·195	1·290	·951	·992	1·287	1·038	·522	·696	·874	·571
5000	„	1·145	1·196	……	·954	·997	1·282	1·038	·520	·694	·872	·577
6000	„	1·143	1·199	……	·958	1·002	1·295	1·038	·517	·693	·870	·582
7000	„	1·144	1·199	……	·961	·999	1·283	1·037	·515	·690	·868	·584
8000	„	1·143	1·200	……	·961	·995	1·288	1·037	·513	·688	·866	·588
10000	„	1·138	……	……	·960	·997	1·302	1·041	·511	·688	·866	·596
12000	„	1·131	……	……	·955	·994	1·282	1·040	·507	·675	·857	·599
15000	„	……	……	……	·947	·995	1·272	1·042	·503	·665	·849	·607
20000	„	……	……	……	·944	1·002	1·262	1·042	·495	·645	·829	·622
25000	„	……	……	……	·940	1·000	……	1·035	·480	·638	·810	·623
28000	„	……	……	……	·936	·993	……	1·035	·472	·643	·814	·622
30000	„	……	……	……	·946	1·005	……	1·050	·475	·643	·825	·630
32000	„	……	……	……	……	1·017	……	1·032	·473	·625	·836	·625
33912	„	……	……	……	·940	……	……	……	……	……	……	·631
Mean Ratios.												
		1·137	1·189	1·282	·946	·993	1·282	1·038	·507	·676	·859	·593
Ratio of Highest to Lowest Ratio.												
		1·022	1·027	1·017	1·039	1·059	1·032	1·010	1·129	1·130	1·115	1·158

TABLE XV.

Ratio of Molecular Volumes of Saturated Vapour to those of Fluorbenzene at corresponding Temperatures.

Temp. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
367·3	1·000	1·118	1·174	1·272	·842	·828	1·336	1·101	1·244	2·076	2·678	·894
382·0	,,	1·130	1·186	1·286	·853	·852	1·329	1·092	1·126	1·805	2·254	·864
393·25	,,	1·135	1·189	1·292	·866	·866	1·322	1·088	1·049	1·635	2·007	·845
410·4	,,	1·150	1·201	1·293	·885	·895	1·316	1·083	·945	1·414	1·714	·818
423·8	,,	1·150	1·203	1·281	·893	·910	1·313	1·078	·875	1·284	1·541	·797
434·85	,,	1·156	1·203	......	·906	·924	1·308	1·073	·826	1·191	1·422	·782
444·25	,,	1·155	1·204	......	·919	·933	1·305	1·067	·788	1·126	1·337	·769
452·8	,,	1·146	1·199	......	·917	·935	1·300	1·059	·755	1·071	1·266	·756
460·4	,,	1·143	1·189	......	·923	·938	1·300	1·054	·729	1·028	1·216	·745
473·6	,,	1·134	......	......	·927	·949	1·300	1·048	·689	·962	1·139	·729
484·95	,,	1·116	......	......	·931	·954	1·291	1·041	·655	·908	1·081	·712
499·7	,,	......	......	......	·926	·959	1·275	1·040	·615	·841	1·016	·694
519·7	,,	......	......	......	·926	·977	1·262	1·030	·572	·763	·943	·681
536·0	,,	......	......	......	·924	·982	......	1·028	·530	·705	·875	·665
544·5	,,	......	......	......	·930	·985	......	1·030	·508	·690	·861	·653
550·0	,,	......	......	......	·940	·995	......	1·041	·497	·676	·864	·649
555·0	,,	......	......	......	......	1·014	......	1·025	·484	·648	·866	·639
559·55	,,	......	......	......	·940	......	......	......	......	......	......	·631
Mean Ratios.												
		1·139	1·194	1·285								
Ratio of Highest to Lowest Ratio.												
		1·036	1·026	1·017	1·116	1·224	1·059	1·074	2·570	3·204	3·110	1·417

TABLE XVI.

Percentage-differences between Highest and Lowest Ratios in each comparison with Fluorbenzene.

	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
Absolute temperatures at corresponding pressures	0·2	0·2	0·2	1·4	2·5	1·0	1·1	10·5	13·8	14·6	6·1
Vapour-pressures at corresponding temperatures…	2·6	2·5	2·8	24·7	44·8	17·4	14·0	391·0	602·0	1035·0	168·5
Molecular volumes of liquid at corresponding pressures	0·3	0·2	0·2	0·6	0·8	1·7	1·5	3·5	4·1	3·6	0·9
Molecular volumes of liquid at corresponding temperatures	0·4	0·3	0·2	1·1	0·9	2·0	2·1	3·2	3·0	3·4	2·3
Molecular volumes of saturated vapour at corresponding pressures	2·2	2·7	1·7	3·9	5·9	3·2	1·0	12·9	13·0	11·5	15·8
Molecular volumes of saturated vapour at corresponding temperatures…	3·6	2·6	1·7	11·6	22·4	5·9	7·4	157·0	220·4	211·0	41·7

The degree of deviation from constancy of the various ratios is indicated by the values given at the foot of each table, representing the ratio of the highest to the lowest ratio in each vertical column. These values are collected together in Table XVI. (p. 252), but for the sake of greater clearness they are given as percentage-differences between the highest and lowest ratios in each case, the lowest ratio being always taken as 100.

It will be seen that the halogen derivatives of benzene show very much smaller deviations from constancy than the other compounds; and I have previously suggested (Trans. Chem. Soc. 1889, p. 486; 1891, p. 125) that the generalizations of Van der Waals do hold good for these bodies, in which case the deviations of the ratios from constancy may be taken as an approximate measure of the experimental errors to be expected with other compounds. It must, however, be admitted that the mean ratios of the absolute temperatures differ sensibly from the mean ratios of the molecular volumes of liquid, though they should be identical if Van der Waals's generalizations were strictly true, while the errors in these determinations are probably very small. Indeed, in the comparison of bromobenzene with fluorbenzene the difference between the mean ratios amounts to 1·5 per cent. (Table XXI.), while the difference between the highest and lowest ratio in either case is only 0·2 per cent. (Table XVI.). It is probable, therefore, that the generalizations of Van der Waals are not quite, though very nearly, true for these bodies.

The deviations from constancy are smallest in the comparisons of the absolute temperatures at corresponding pressures, and of the molecular volumes of liquid at corresponding pressures and temperatures; and this is no doubt to be expected, since the ratio of the highest to the lowest absolute temperature or volume of liquid for any one substance does not amount to 3 : 1, whereas the ratio of the highest to the lowest pressure is from 1700 to 10000 : 1, and of the highest to the lowest molecular volume of saturated vapour from 3·7 : 1 in the case of iodobenzene, where the range is limited, to 2670 to 1 with acetic acid. It is also impossible to determine the volumes of saturated vapour with anything like the same degree of accuracy as the volumes of liquid, the liability to error by the new method being especially great at the lowest temperatures.

The eleven substances which are compared with fluorbenzene may be arranged in three groups as below:—

1. *Chlorobenzene*, *Bromobenzene*, *Iodobenzene*.—As already pointed out, the generalizations of Van der Waals are very nearly true for these bodies when compared with fluorbenzene.

2. *Benzene*, *Carbon tetrachloride*, *Stannic chloride*, *Ether*.—With these substances the generalizations may be taken as rough approximations to the truth, but the deviations of the ratios from constancy are in most cases much too large to be attributed to error of experiment.

3. *The three Alcohols and Acetic Acid*.—The majority of the generalizations do not hold good at all; the deviations of the ratios from constancy are, however, not very great in the case of the molecular volumes of liquid at corresponding pressures and temperatures.

Discussion of the Generalizations.

1. *Corresponding Temperatures and Pressures*.—It must, I think, be concluded that the statement that "if the absolute temperatures of various substances are proportional to their absolute critical temperatures their vapour-pressures will be proportional to their critical pressures" has not been proved by experiment to be true except in a very limited number of cases; indeed, when the alcohols and acetic acid are compared with fluorbenzene, the statement is quite wide of the mark. It follows, therefore, that in the comparison of the molecular volumes the ratios at corresponding pressures must differ from those at corresponding temperatures.

2. *Molecular Volumes of Liquid*.—The deviations of the ratios from constancy are smaller in this case than in any of the others, but are certainly not within the limits of experimental error. The comparisons at corresponding pressures seem to be somewhat better on the whole than at corresponding temperatures, but the difference is not very marked.

3. *Molecular Volumes of Saturated Vapour.*—At corresponding pressures the deviations from constancy are within the limits of experimental error in the case of ether and stannic chloride, and are relatively small for benzene and carbon tetrachloride.

There can be no doubt that with the saturated vapours the comparisons at corresponding pressures are very much better than at corresponding temperatures; and it may therefore be concluded that it is better to compare the molecular volumes both of liquid and of saturated vapour at corresponding pressures than at corresponding temperatures.

Saturated Vapours.

It has been pointed out (Trans. Chem. Soc. 1891, p. 137) that if the generalizations of Van der Waals were strictly true, it would follow that the ratios of the actual densities of the saturated vapours of different substances to their theoretical densities should be equal at corresponding pressures; and as this method of comparison is a very convenient one I have thought it worth while to tabulate the ratios, although it has already been shown that the relation cannot be strictly true. The values for benzene and its halogen derivatives have already been published (*loc. cit.*); but there are one or two obvious small irregularities, and I have therefore mapped the ratios against temperature and constructed curves for each substance (Plate VII.). The smoothed values read from the curves are given in this paper, and the molecular volumes of saturated vapour have also been recalculated from the density ratios. The same method of procedure has been adopted with the other substances.

The close similarity in the behaviour of the halogen derivatives of benzene and the approximate agreement in the case of the members of Group II. is very clearly shown in Table XVII.

It is generally admitted that gaseous acetic acid at low temperatures contains molecules more complex than those corresponding to the ordinary formula $C_2H_4O_2$, and from the

TABLE XVII.

Ratios of Actual to Theoretical Densities of Saturated Vapour at corresponding Pressures.

Pressure. C_6H_5F.	C_6H_5F.	C_6H_5Cl.	C_6H_5Br.	C_6H_5I.	C_6H_6.	CCl_4.	$SnCl_4$.	$(C_2H_5)_2O$.	CH_3OH.	C_2H_5OH.	C_3H_7OH.	CH_3COOH.
20	......	......	......	......	......	......	......	......	......	......	......	1·907
50	......	......	......	......	......	......	......	......	......	......	......	1·829
100	......	......	......	......	......	......	......	......	1·022	1·004	......	1·770
200	......	......	......	......	......	......	......	......	1·032	1·010	......	1·718
400	......	......	......	......	......	......	......	1·029	1·050	1·019	1·025	1·678
760	......	......	......	......	......	......	1·040	1·046	1·071	1·033	1·047	1·657
1000	1·037	1·047	1·063	1·053	1·037	1·047	1·047	1·054	1·084	1·042	1·060	1·654
1500	1·056	1·059	1·077	1·065	1·052	1·059	1·063	1·072	1·105	1·059	1·081	1·659
2000	1·073	1·072	1·091	1·079	1·066	1·073	1·079	1·090	1·122	1·075	1·099	1·669
3000	1·107	1·099	1·116	1·107	1·091	1·098	1·107	1·122	1·157	1·103	1·132	1·690
4000	1·138	1·127	1·142	1·136	1·112	1·122	1·136	1·153	1·189	1·129	1·164	1·713
5000	1·166	1·153	1·168	......	1·136	1·147	1·162	1·180	1·219	1·153	1·192	1·735
6000	1·193	1·181	1·192	......	1·159	1·173	1·187	1·207	1·249	1·177	1·217	1·759
7000	1·217	1·206	1·217	......	1·180	1·197	1·213	1·233	1·277	1·203	1·242	1·781
8000	1·247	1·233	1·242	......	1·206	1·233	1·238	1·260	1·306	1·230	1·267	1·805
10000	1·300	1·291	......	......	1·260	1·275	1·287	1·308	1·360	1·275	1·315	1·852
12000	1·349	1·347	......	......	1·316	1·329	1·343	1·358	1·419	1·343	1·373	1·904
15000	1·431	......	......	......	1·408	1·410	1·435	1·436	1·507	1·437	1·462	1·983
20000	1·600	......	......	......	1·582	1·571	1·619	1·605	1·700	1·643	1·664	2·153
25000	1·818	......	......	......	1·806	1·790	......	1·837	1·985	1·876	1·927	2·434
28000	2·011	......	......	......	2·006	1·994	......	2·032	2·228	2·054	2·116	2·691
30000	2·206	......	......	......	2·180	2·164	......	2·199	2·420	2·250	2·282	2·920
32000	2·450	......	......	......	......	2·376	......	2·485	2·699	2·561	2·504	3·252
33912	4·400	(4·43)	(4·46)	(4·43)	4·370	(4·30)	(4·35)	(4·40)	(5·18)	(4·65)	(4·70)	5·792

The bracketed values are calculated from the molecular critical volumes given in Table XXII.

very high density at the critical point it would appear that many of these complex molecules have escaped dissociation even at this high temperature.

It will be seen that with the alcohols the differences from the other substances become generally more marked as the critical point is approached. At the highest pressure the vapour-density of methyl alcohol is very distinctly higher than that of any member of the first or second group at the corresponding pressure; and this fact appears to favour the conclusion of M. Guye and others, that some of the molecules of methyl alcohol at the critical point and in the liquid state at all temperatures are more complex than in the ordinary gaseous state. On the other hand, by a comparison of the densities of the saturated vapours of acetic acid with those of the alcohols and ether, Dr. Ramsay and I were led to the conclusion that the molecules of ordinary substances, including the alcohols, are not more complex in the liquid than in the gaseous state. This conclusion, so far as the members of Groups I. and II. are concerned, is strengthened by the results given in the preceding Table; but it is certainly weakened to some extent in the case of the alcohols, or at any rate of methyl alcohol.

There is no doubt that these bodies do show marked differences in many of their properties from the majority of compounds; and the most plausible explanation of these differences seems to be the existence of complex molecules in the liquid state, although there is ample proof that they are not present in the saturated vapours at low temperatures.

As the alcohols differ so widely from other substances, it is of interest to find whether the relations of Van der Waals hold good when they are compared among themselves. I have therefore calculated the ratios of the absolute temperatures, pressures, and volumes of ethyl and propyl alcohol to methyl alcohol and of propyl alcohol to ethyl alcohol. The results are given in the following Tables:—

TABLE XVIII.—Alcohols at corresponding Pressures.

Pressures of C_6H_5F.	Absolute Temperatures.			Molecular Volumes of Liquid.			Molecular Volumes of Vapour.		
	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.
20	1·0363	1·0860	1·0479	......	......	1·2998	......	......	......
50	1·0327	1·0785	1·0443	1·4565	1·8918	1·2989	.....	......	......
100	1·0302	1·0741	1·0427	1·4597	1·8960	1·2989	1·310	......	......
200	1·0274	1·0690	1·0405	1·4562	1·8903	1·2981	1·311	......	......
400	1·0238	1·0632	1·0385	1·4546	1·8866	1·2971	1·317	1·707	1·296
760	1·0205	1·0581	1·0368	1·4535	1·8840	1·2962	1·321	1·697	1·284
1000	1·0187	1·0561	1·0367	1·4527	1·8831	1·2962	1·323	1·693	1·279
1500	1·0169	1·0533	1·0358	1·4515	1·8812	1·2960	1·321	1·688	1·277
2000	1·0186	1·0541	1·0349	1·4430	1·8806	1·3033	1·328	1·687	1·271
3000	1·0139	1·0465	1·0322	1·4493	1·8765	1·2977	1·328	1·676	1·262
4000	1·0125	1·0450	1·0321	1·4534	1·8838	1·2959	1·332	1·673	1·256
5000	1·0113	1·0461	1·0344	1·4555	1·8843	1·2946	1·335	1·677	1·256
6000	1·0106	1·0454	1·0344	1·4543	1·8844	1·2958	1·339	1·682	1·256
7000	1·0100	1·0447	1·0344	1·4558	1·8864	1·2957	1·340	1·685	1·257
8000	1·0097	1·0442	1·0341	1·4538	1·8874	1·2983	1·339	1·687	1·260
10000	1·0094	1·0440	1·0343	1·4543	1·8864	1·3001	1·345	1·692	1·259
12000	1·0090	1·0441	1·0348	1·4541	1·8933	1·3021	1·331	1·683	1·271
15000	1·0084	1·0443	1·0356	1·4534	1·8974	1·3054	1·321	1·688	1·278
20000	1·0093	1·0465	1·0368	1·4572	1·8978	1·3023	1·304	1·679	1·285
25000	1·0070	1·0455	1·0382	1·4576	1·9013	1·3044	1·330	1·688	1·269
28000	1·0061	1·0458	1·0394	1·4466	1·8953	1·3102	1·363	1·687	1·266
30000	1·0061	1·0464	1·0400	1·4540	1·8994	1·3063	1·352	1·736	1·284
32000	1·0063	1·0467	1·0402	1·4482	1·8897	1·3049	1·324	1·768	1·335
33912	1·0060	1·0462	1·0399	......	......	......	......	......	......
Mean Ratios.				1·4534	1·8890	1·2999	1·329	1·693	1·274
Ratio of Highest to Lowest Ratio.	1·030	1·040	1·015	1·012	1·013	1·012	1·040	1·057	1·063

TABLE XIX.—Alcohols at corresponding Temperatures.

Absolute Temperatures of C_6H_5F.	Vapour-Pressures.			Molecular Volumes of Liquid.			Molecular Volumes of Saturated Vapour.		
	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.	$\frac{C_2H_5OH}{CH_3OH}$.	$\frac{C_3H_7OH}{CH_3OH}$.	$\frac{C_3H_7OH}{C_2H_5OH}$.
272·25	......	......	....	......	......	......	......	......	......
289·3	......	·276	......	......	......	......	......	......	......
303·9	·483	·320	·662	......	......	1·2968	......	......	......
320·25	·513	·370	·720	1·4443	1·8730	1·2968	......	......	......
338·75	·557	·426	·765	1·4444	1·8746	1·2978	1·851	......	......
358·1	·602	·476	·791	1·4439	1·8735	1·2975	1·718	......	......
367·3	·623	·502	·805	1·4435	1·8734	1·2978	1·669	2·153	1·290
382·0	·653	·538	·824	1·4431	1·8734	1·2981	1·603	2·002	1·249
393·25	......	......	·829	1·4430	1·8734	1·2983	1·560	1·914	1·227
410·4	·705	·593	·841	1·4422	1·8743	1·2995	1·496	1·813	1·212
423·8	·721	·607	·842	1·4380	1·8753	1·3041	1·471	1·760	1·199
434·85	·738	·622	·842	1·4407	1·8755	1·3030	1·442	1·722	1·194
444·25	·747	·630	·845	1·4433	1·8807	1·3031	1·429	1·696	1·187
452·8	·757	·639	·844	1·4465	1·8843	1·3027	1·419	1·677	1·182
460·4	·765	·643	·841	1·4478	1·8854	1·3023	1·409	1·667	1·183
473·6	·771	·643	·834	1·4489	1·8922	1·3060	1·396	1·654	1·184
484·95	·775	·652	·841	1·4485	1·8964	1·3092	1·387	1·652	1·191
499·7	·781	·651	·834	1·4476	1·9015	1·3135	1·368	1·652	1·208
519·7	·792	·649	·820	1·4478	1·9052	1·3160	1·334	1·648	1·236
536·0	·795	·644	·810	1·4509	1·9058	1·3104	1·329	1·650	1·242
544·5	·797	·640	·803	1·4455	1·9010	1·3151	1·327	1·656	1·247
550·0	·800	·644	·787	1·4475	1·8946	1·3089	1·360	1·738	1·278
555·0	·800	·635	·794	1·4438	1·8758	1·2992	1·338	1·786	1·335
559·55	·801	·638	·797	......	......	......	......	......	......
Ratio of Highest to Lowest Ratio.	1·658	2·362	1·276	1·009	1·018	1·015	1·395	1·306	1·129

The percentage difference between the highest and lowest ratios in the various comparisons—including those with fluorbenzene—are given in the following Table:—

TABLE XX.

	$\frac{CH_3OH}{C_6H_5F}$	$\frac{C_2H_5OH}{C_6H_5F}$	$\frac{C_3H_7OH}{C_6H_5F}$	$\frac{C_2H_5OH}{CH_3OH}$	$\frac{C_3H_7OH}{CH_3OH}$	$\frac{C_3H_7OH}{C_2H_5OH}$
Absolute temperatures at corresponding pressures.........	10·5	13·8	14·6	3·0	4·0	1·5
Vapour-pressures at corresponding temperatures......	391·0	602·0	1035·0	65·8	136·2	27·6
Molecular volumes of liquid at corresponding pressures	3·5	4·1	3·6	1·2	1·3	1·2
Molecular volumes of liquid at corresponding temperatures	3·2	3·0	3·4	0·9	1·8	1·5
Molecular volumes of saturated vapour at corresponding pressures	12·9	13·0	11·5	4·0	5·7	6·3
Molecular volumes of saturated vapour at corresponding temperatures	157·0	220·4	211·0	39·5	30·6	12·9

When the alcohols are compared with one another, the deviations of the ratios from constancy are much smaller than when fluorbenzene is taken as the standard substance, but they are still far outside the limits of experimental error. It may, perhaps, be said—as with the members of Group II. in the comparisons with fluorbenzene—that the generalizations offer a rough approximation to the truth.

Here, again, the comparison of the molecular volumes of saturated vapour at corresponding pressures is very much more satisfactory than at corresponding temperatures.

In the Philosophical Magazine for November 1890, p 417, it was pointed out by Prof. Orme Masson that the ratio of the molecular volumes (in the liquid state) of any two members of certain groups of nearly related carbon compounds, at their boiling-points under equal pressure, is equal to the ratio of those boiling-points, expressed on the absolute scale of temperature, or $\frac{V}{V'} = \frac{T}{T'}$.

In the same number of the Philosophical Magazine (p. 423)

I showed that Masson's relation is a special case of a more general one which should hold good if the generalizations of Van der Waals were strictly true. This relation may be expressed by the equations

$$\frac{v}{v'}=\frac{V}{V'}=\frac{T}{T'}\times\frac{p'}{p},$$

where v and v' are the molecular volumes of saturated vapour, V and V′ those of liquid, and T and T′ the boiling-points on the absolute scale of any two substances at corresponding pressures p and p'.

When the critical pressures are equal $p=p'$ and the equations become

$$\frac{v}{v'}=\frac{V}{V'}=\frac{T}{T'},$$

or the molecular volumes, whether of liquid or saturated vapour, at the boiling-points under equal pressure are proportional to the boiling-points expressed on the absolute scale.

That the relation cannot be strictly true when the critical pressures are different is obvious, since it is not generally true that the vapour-pressures at corresponding temperatures are corresponding pressures. It is still possible, however, that the relation may be true for the critical temperatures, pressures, and volumes. Unfortunately, the critical volumes have only been determined in a few cases, but it has been shown that the ratios of the molecular volumes at corresponding pressures do not vary within very wide limits; and it is, therefore, possible to test the relation in a limited form, taking the critical temperatures and pressures and the mean ratios of the molecular volumes at corresponding pressures.

A preliminary comparison was given in the paper referred to, but it was very incomplete, and there was a small error in some of the calculations owing to the misplacement of a figure in the critical pressure of fluorbenzene, which was given as 33190 instead of 33910.

As the experimental results have been considerably extended since that time and now include the molecular volumes of the saturated vapours, it is possible to give the Table in a complete form.

TABLE XXI.

Comparison of Values of $\frac{T}{T'} \times \frac{p'}{p}$ with Mean Ratios of Molecular Volumes at corresponding Pressures.

Substances compared.	$\frac{T}{T'} \times \frac{p'}{p}$.	Mean Ratios of Molecular Volumes.		Percentage difference between highest and lowest of the three values.
		Liquid.	Saturated Vapour.	
C_6H_5Cl/C_6H_5F	1·1308	1·1246	1·137	1·1
C_6H_5Br/C_6H_5F	1·1974	1·1802	1·189	1·5
C_6H_5I/C_6H_5F	1·2884	1·2772	1·282	0·9
C_6H_5Br/C_6H_5Cl	1·0589	1·0495	1·047	1·1
C_6H_5I/C_6H_5Cl	1·1394	1·1358	1·132	0·7
C_6H_5I/C_6H_5Br	1·0760	1·0822	1·080	0·6
C_6H_6/C_6H_5F	·9351	·9439	·946	1·2
CCl_4/C_6H_5F	·9861	1·0195	·993	3·4
$SnCl_4/C_6H_5F$	1·2771	1·2793	1·282	0·4
$(C_2H_5)_2O/C_6H_5F$	1·0468	1·0342	1·038	1·2
CH_3OH/C_6H_5F	·5203	·4228	·507	23·1
C_2H_5OH/C_6H_5F	·6537	·6153	·676	9·9
C_3H_7OH/C_6H_5F	·8533	·7995	·859	7·4
CH_3COOH/C_6H_5F	·8303	·6342	·593	40·0
C_2H_5OH/CH_3OH	1·2564	1·4534	1·329	15·7
C_3H_7OH/CH_3OH	1·6401	1·8890	1·693	15·2
C_3H_7OH/C_2H_5OH	1·3054	1·2999	1·274	2·5

For the halogen derivatives of benzene $p=p'$.

It will be seen that the differences between the three values are not very great, except in the comparisons of the three alcohols and acetic acid with fluorbenzene, and of ethyl and propyl alcohol with methyl alcohol. In the other cases, including the comparison of propyl with ethyl alcohol, the relation

$$\frac{v}{v'} = \frac{V}{V'} = \frac{T}{T'} \times \frac{p'}{p}$$

may be considered as approximately true with the limitations already stated.

The two substances—methyl alcohol and acetic acid, and to a small extent ethyl and propyl alcohol—are clearly exceptional in their behaviour, and it is of interest to note, in comparing these substances with fluorbenzene, that while with acetic acid the ratios of the molecular volumes both of liquid and saturated vapour are very low, with methyl alcohol it is only in the liquid state that this is to be observed. This may again point to the existence of complex molecules of acetic acid in both the liquid and gaseous states, but of methyl alcohol in the liquid state only, except possibly very near the critical point.

With ethyl and propyl alcohols the ratios of the molecular volumes of liquid to those of fluorbenzene are also low, though not nearly to the same extent as with methyl alcohol.

Note on the Determination of Critical Constants.

Of the three critical constants, temperature, pressure, and volume, the first is the most easily determined, for by employing as heating-agents the vapours of pure liquids boiling under known pressures, the temperature is perfectly under control and is easily measured. Moreover, the presence of a very small amount of impurity does not influence the critical temperature of a substance to nearly the same extent as the critical pressure.

The critical pressure may also as a rule be determined without very much difficulty, provided that absence of impurity is ensured, but this point is of the utmost importance. In the case of substances that attack mercury at high temperatures, such as carbon tetrachloride and stannic chloride, the ordinary method of operation requires modification, and the calculations become more laborious, but otherwise the difficulty is not greatly increased.

The determination of the critical volume of a substance, even when perfectly pure and without action on mercury, is, however, a much more difficult matter. That this is so is evident from the form of the curves representing the relation of the volumes of liquid and of saturated vapour to the temperature or pressure, as shown in the accompanying diagrams (p. 264) constructed from the results with ethyl alcohol.

A very small alteration of temperature, such as 0·1°, at or just below the critical point, produces a considerable alteration in the volume ; therefore in order to obtain a direct reading of the critical volume it is necessary that the substance shall be exactly at its critical temperature.

I have assumed that a substance is in this state when on rapidly increasing the volume somewhat above the critical volume the fall of temperature due to the expansion causes a momentary separation of liquid and vapour.

In order to determine the critical volume, I note the position of this temporary mark of division and then diminish the volume slightly. After waiting a few minutes for the temperature to become constant again, I increase the volume very slightly but rapidly, and again note the position of the mark of division, which is now nearer to the top of the tube. Proceeding in this way it is possible under favourable conditions to make the substance occupy such a volume that a very slight but rapid expansion gives a temporary mark of division of liquid and vapour almost exactly at the top of the tube. This volume I take to be the critical volume, and I have succeeded in determining it directly in the case of benzene, fluorbenzene, and acetic acid; while with chlorobenzene, for which mercury vapour was employed as a jacket, it was only possible to obtain a rough approximation to the true volume on account of the slight unsteadiness of the temperature.

Fig. 1.

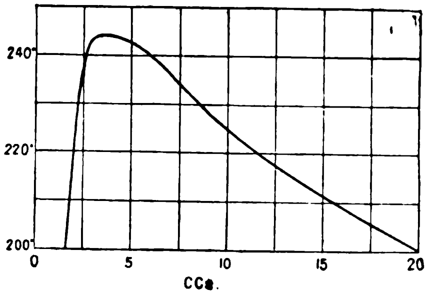
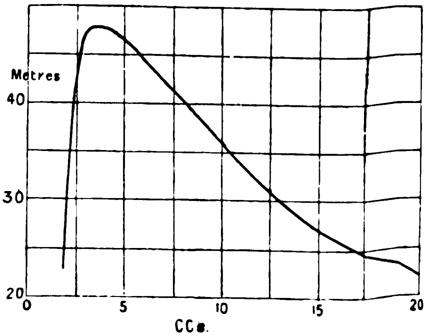


Fig. 2.



The critical volumes of the other substances may probably be ascertained with fair accuracy in the following manner :—

At low temperatures and pressures the ratios of the molecular volumes of liquid and saturated vapour of any one substance to those of fluorbenzene at corresponding temperatures and pressures differ somewhat widely as a rule, but as the critical point is approached the differences diminish and at the critical point itself all four values should, of course, be identical. It follows, therefore, that by mapping the ratios against temperature, four straight lines or curves should be obtained, which, when produced, should cut one another at the critical temperature, and the point of intersection should give the ratio of the molecular critical volume to that of fluorbenzene.

It will be seen from the accompanying diagrams, figs. 3 to 6, Plates VIII. & IX., that in the case of benzene and acetic acid the four curves do very nearly cut one another at the critical temperature, and the ratios of the critical volumes thus obtained agree very well with those calculated from the experimental results. The other substances will be considered in their order.

Halogen Derivatives of Benzene.—Accurate determinations of the volumes of liquid have been obtained only up to 280°, but the ratios are very nearly constant and are practically identical at corresponding temperatures and pressures; the extrapolation may therefore be considered justifiable. In the case of chlorobenzene a few approximate determinations have been made near the critical temperature; they give slightly higher ratios than those at lower temperatures, and agree rather more closely with the mean ratio of the absolute temperatures at corresponding (equal) pressures.

The molecular volumes of the saturated vapours give generally higher ratios than those of the liquid; this may be partly due to experimental error; but as the deviations are nearly all in the same direction, it seems hardly justifiable to attribute them entirely to this cause. It is evidently impossible to make use of these ratios in determining the critical volumes; but the mean ratios of the molecular volumes of liquid may probably be relied on to give fairly accurate results (Trans. Chem. Soc. lv. p. 517).

Carbon Tetrachloride.—As the observations extend to within a few degrees of the critical temperature, there does not seem to be much room for error.

Stannic Chloride.—The molecular volumes of saturated vapour cannot be made use of, as the results at the highest temperatures are a little doubtful owing to slight decomposition of the substance ; considerable extrapolation would also be necessary. The ratios for the liquid state give, however, very nearly straight lines, which may therefore probably be extrapolated without much error.

Ether.—The four curves would evidently be very close together at the critical temperature, but as the results with the saturated vapour are somewhat irregular it seems best to rely only upon those with the liquid.

Methyl Alcohol.—In this case the curves for the liquid state do not seem to agree well with those for the saturated vapour; I have taken the mean value, but it is evidently open to some doubt.

Ethyl and Propyl Alcohols.—The agreement is better with these alcohols, but the volumes of saturated vapour are somewhat irregular near the critical point.

Acetic Acid.—The four curves meet very satisfactorily at the critical temperature.

The mean ratios of the molecular volumes to those of fluorbenzene are given in the following Table, together with the molecular and specific critical volumes calculated from them,

TABLE XXII.

Substance.	Mean Ratio.	Molecular Volume. Calculated.	Specific Volume.		
			Calculated.	Observed.	Ramsay and Young.
C_6H_5F	......	... ..	......	2·43	......
C_6H_5Cl	1·1246	262	2·34	2·34–2·45	......
C_6H_5Br	1·1802	275	1·76	.....	......
C_6H_5I	1·2772	298	1·47	......	......
C_6H_6	·944	220	2·83	2·82	......
CCl_4	1·010	235	1·53	......	......
$SnCl_4$	1·290	301	1·16	......	......
$(C_2H_5)_2O$	1·047	244	3·30	......	4·06
CH_3OH	·440	103	3·21	......	3·7
C_2H_5OH	·620	144	3·15	......	3·5
C_3H_7OH	·800	186	3·11	......	3·6
CH_3COOH	·630	147	2·45	2·46	......

taking the observed critical volume of fluorbenzene as correct. For the sake of comparison the specific volumes directly observed and also those previously adopted by Ramsay and myself are given.

It will be seen that the values for benzene and acetic acid, calculated from the mean ratios, agree very well with those directly observed, and this may be taken as evidence in favour of the accuracy of the method of direct measurement adopted. On the other hand, the calculated values for ether and the alcohols are considerably lower than those previously given by Ramsay and myself. But it may be pointed out that the critical temperature of ether was almost certainly slightly underestimated, owing to the employment of the vapour of methyl salicylate as a jacket. There is no reason to suspect any error in the determination of the critical pressure; and if the constants for Biot's formula be taken as correct, the calculated temperature corresponding to the critical pressure would be 194·4 instead of 193·8, and the higher value agrees well with a more recent determination by Ramsay, who in this case employed the vapour of quinoline as a jacket. The correction of the critical temperature of ether would give a lower value to the critical volume.

Again, Professor Tait has recently made an exhaustive mathematical investigation of the relations of pressure, temperature, and volume of several substances, including ether, and he has pointed out that the critical temperature is probably a little higher than 193·8, and that the critical volume of this substance is certainly lower than 4 cubic centim., and is more probably about 3·5 cubic centim., a value that agrees fairly well with that calculated by the method described.

In the case of the alcohols the critical volumes given by Ramsay and myself were estimated by inspection of the curves representing the relation of the specific gravities of liquid and saturated vapour to the temperature, and of the specific volumes to the pressures, and they were admittedly only approximate values.

M. Guye has shown (*Comptes Rendus,* cxii. p. 1257) that an approximate relation exists between the critical temperature, pressure, and volume of a substance and its theoretical vapour-density compared with air. This is expressed by the equations

$$d = 1146 \frac{\delta\theta}{\pi(1070+\theta)} = \frac{\mathrm{M}}{28{\cdot}87};$$

where δ is the specific gravity at the critical point compared with water at 4°, θ the absolute critical temperature, π the critical pressure, and M the molecular weight. The critical volumes of ether and the alcohols previously given by Ramsay and myself are certainly in better agreement with this relation than those which I now find; but it may be mentioned that, with the exception of iodobenzene, all the substances referred to in this paper give somewhat higher values of d than the quotient $\frac{\mathrm{M}}{28{\cdot}87}$ if the molecular volumes in Table XXII. be taken as correct.

It may, I think, be safely stated that the true critical volumes are not lower than those directly observed, but it is possible that they may be a little higher. It is therefore perhaps too much to say that the values given in the table are definitely established, but I am inclined to think that they may be accepted as fairly close approximations to the truth.

If that is so, it would appear from Table XVII. that *the ratio of the actual critical density to the theoretical density is for very many substances about* 4·4. Ethyl and propyl alcohol give somewhat higher values, while for methyl alcohol and acetic acid, especially the latter, the ratios are extremely high.

Prof. Ramsay said the results proved that Van der Waals's generalizations were only rough approximations, and he suggested that some force had been neglected, or a term omitted from the equations. Perhaps the assumption that the molecules are incompressible was not correct. He also strongly protested against the tacit assumption of the Van der Waals's laws, and deductions made therefrom, which had recently become so common, particularly in German text-books.

Prof. Perry inquired whether the quantities a, b, and a had been determined for different substances and found to be constant.

Prof. Ramsay said that for substances in states analogous to those of perfect gases the quantities were approximately constant, but when the liquid state was approached this was no longer true. According to Prof. Tait the two states were not continuous.

Prof. Herschel remarked that Prof. Tait had established his law on the assumption that the co-volume is four times the volume occupied by the molecules. This law, he said, had been amply verified by experiments on explosions.

Dr. Burton, referring to Prof. Ramsay's remarks on the compressibility of molecules, said the law of force between attracting molecules should be accurately known before any deductions were made; and he pointed out that at constant volume the pressure should be proportional to the absolute temperature if allowance be made for the negative pressure of attraction.

Mr. Blakesley, in speaking of molecular forces, said he had observed that when water is allowed to evaporate from glass, a furrow is formed in the glass which marks out the original boundary of the liquid. To all appearances the particles of glass are torn away by the molecular forces acting along the boundary.

XXIX. *The Construction of Non-Inductive Resistances.*

By Prof. W. E. AYRTON, *F.R.S.*, *and* T. MATHER*.

[Plate X.]

WITH all electric methods devised for measuring the power given by a varying current to a circuit that may possess inductance or capacity it is necessary to employ a resistance which shall have zero inductance, consequently it is important to consider how such resistances may most easily be made.

If the inductance of this nominally non-inductive resistance be not zero, the error thus introduced into the measurement can still be made relatively unimportant if the time-constant of this portion of the circuit be made small. For, as shown by Dr. Sumpner and one of us, in a paper[*] read before this Society on June 12th of this year, the following proportion holds true in all the nine methods of measuring power there considered:—

---

* Read June 26, 1891.

$$\frac{\text{the watts as measured}}{\text{the true watts}} = \frac{1 + \tan\theta \, . \tan\phi}{1 + \tan^2\phi},$$

where θ and ϕ are the angles of phase-difference between the current and the P.D. for the circuit the power given to which we desire to measure, and for the auxiliary circuit respectively.

Now for any given configuration of a circuit, the time-constant will be the smaller the higher the specific resistance of the conductive material; hence it is very desirable to use a material of high specific resistance, like carbon. For this reason, glow-lamps constitute valuable small time-constant resistances, but carbon has the disadvantage that its resistance varies rapidly with temperature. Hence, since with the methods of measuring power referred to it is necessary to know the resistance of the non-inductive circuit at the moment of making the measurement, it follows that if carbon be employed an extra measurement has to be made.

When the non-inductive resistance is put in series with the circuit the power given to which we desire to measure, as, for example, with the three-voltmeter method of measuring power, a measurement of the resistance of the glow-lamps merely means the reading of an extra instrument at the moment the power-test is made; but when the two circuits are joined in parallel, as, for example, with the one-voltmeter and two-ammeter method of measuring power, a measurement of the resistance of the glow-lamps cannot be made simultaneously with the power-test, since to do this would require the introduction of an ammeter, and therefore of inductance, into the circuit, which should be non-inductive.

Further, carbon is unsuitable for portable resistances on account of its brittle nature. Platinoid, on the other hand, is flexible, has a low temperature-coefficient, and a high specific resistance, though not of course nearly as high as

---

[*] "Alternate Current and Potential Difference Analogies in the Methods of Measuring Power," *ante*, p. 172.

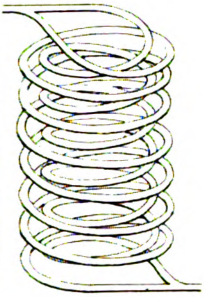


Fig. 2.

Right and Left Handed Spirals in Parallel. Opposing one another.

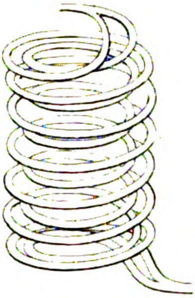


Fig. 4.

Two Right Handed Spirals in Parallel, Assisting one another.

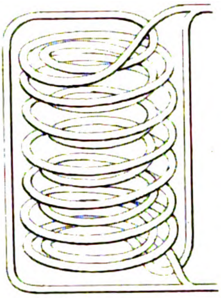


Fig. 5.

Two Right Handed Spirals in Parallel, Opposing one another.

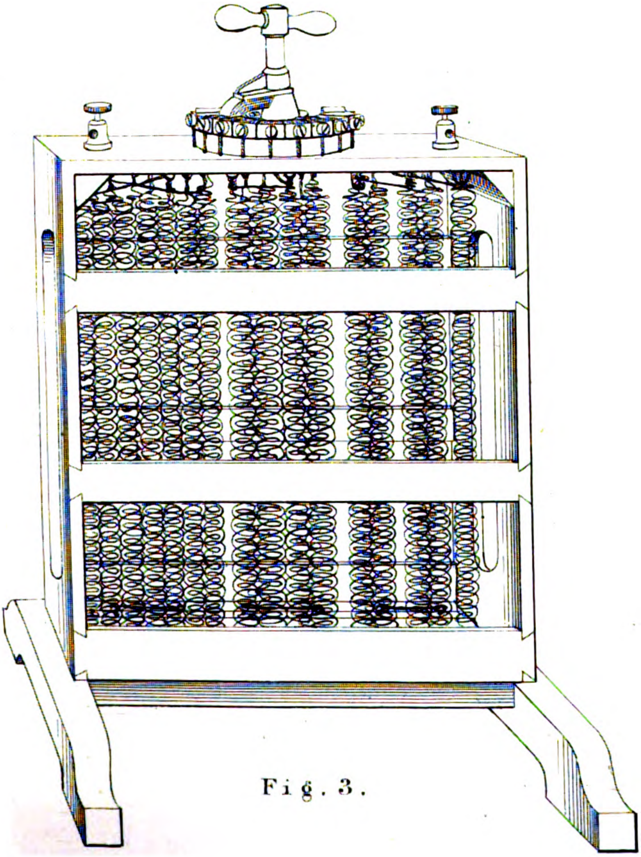


Fig. 3.

Mintern Bros lith.

that possessed by carbon. Platinoid therefore appeared to us to be the best material to employ in the construction of non-inductive resistances to be used for power-tests.

It is well known that a wire doubled on itself has a very small inductance, which approximates to 3·77 times the total length of the wire in centimetres as the parts approach each other. This value can, as Maxwell pointed out, be reduced by using flat strips instead of round wires; and, if the strips be bare and be placed vertically, it is clear that the cooling action will be considerable, so that relatively strong currents will produce but little rise in temperature or increase of resistance.

We therefore decided in 1887 to construct the non-inductive resistance seen in fig. 1, Plate X., consisting of twelve platinoid strips, each 6 metres long, 4 centim. wide, and 0·25 millim. thick. Each strip is doubled on itself, two layers of carefully shell-lacqued silk, 0·075 millim. thick, being inserted between the front and back portions. The whole is bound together by means of a narrow silk ribbon wrapped round spirally, considerable gaps being left between the spires of the silk ribbon so that the platinoid should have plenty of free surface for cooling.

For the purpose of expelling moisture from the silk and the shell-lac varnish, when the resistance-strips are first put up, a current was passed through each strip strong enough to make it fairly hot. During this heating frequent short-circuitings occurred from rough points of the metal piercing the silk, a single layer only of which was originally employed to separate the front and back portions. An additional layer of silk was therefore inserted. Trouble was also experienced from the rough edges of the metal causing short-circuits; but by cutting the silk wider than the metal and by folding the edge of the silk over the edge of the platinoid strip, this difficulty was overcome.

The doubled strips are permanently joined up in sets of three, and to the ends of each set are soldered mercury-cups and binding-screws. The four sets of three can be joined up in series, or in parallel, or in parallel-series by bridge-pieces dipping into the mercury-cups, and, when all are in series, the resistance of the 72 metres of strip at 15° C. is 2·932 ohms. The resistance of the arrangement does not alter by more than one tenth per cent. when a current of 15 amperes is passing through each strip.

The wooden frame which carries this series of platinoid strips hangs on paraffined ebonite pegs attached to one of the laboratory walls, so that it is well insulated.

The inductance of the set of strips, even when all are in series, is so small that we have not been able to measure it even with the secohmmeter. Some of the tests have given indications of a small negative result, which, if true, would mean that the capacity-effect slightly overbalanced the inductance; but a calculation, which we have made, appears to show that with the dimensions in question such a result is impossible. We can therefore only conclude that this resistance-frame, which was constructed by two of our former students, Messrs. C. G. Lamb and E. W. Smith, fulfils the object for which it was intended so well that the inductance cannot be detected with certainty by any test that we have hitherto tried. We therefore have decided to duplicate the arrangement, the wooden framework (only one half of which is seen in fig. 1) having been constructed large enough to hold a second set of strips.

Another method of constructing non-inductive resistances for large currents, which has been in use at the Central Institution for the past eighteen months, is illustrated in fig. 2, and consists in winding two bare platinoid wires of equal length and thickness into two spirals, one right-handed, the other left-handed, the diameters of the two spirals differing slightly from each other so that one spiral can be placed inside the other. They are then connected up in parallel, so that when a current is sent through them it circulates clockwise round one spiral and counterclockwise round the other, the magnetic effects of the two thus tending to neutralize one another and to produce a combination with small inductance.

The inductance of coils constructed in this way is almost as small as if the wires were doubly wound like an ordinary resistance-coil, *but they possess the great advantage that parts differing much in potential are not close to one another.* The wires therefore require no insulating covering, for no harm will occur if one spiral accidentally touches the other, provided that reasonable care has been taken to space the convolutions fairly uniformly. Further, as the cooling surface, for a given total cross section of the conductor, is much greater for two concentric spirals than for a single spiral of thicker wire, it follows that this method of constructing a resistance not only greatly reduces the inductance, but also enables a much larger current to be carried for a given variation of resistance produced by the current. Hence for a given current and for a given permissible rise of temperature finer wire can be used; the coils can therefore be made shorter, and the inductance for this reason still further lessened.

Lastly, even when such right- and left-handed spirals are traversed by a direct current, much less disturbance is produced in a neighbouring delicate galvanometer than if a singly-wound spiral resistance be employed; so that these right- and left-handed spirals of bare wire are valuable in the construction of resistance-frames for large direct as well as for large alternating currents.

A number of portable resistance-coils have been constructed in this way for general use in the laboratories of the Central Institution. One of these is seen in fig. 3. It has also been found convenient to fit up a number of stationary right- and left-handed concentric spirals of platinoid in the dynamo laboratory, for use in the regular experiments with alternate currents.

Table I. gives the particulars of some portable sets of stand coils in the laboratories of the Central Institution, the second three of which have been composed of right- and left-handed spirals in the way just described and illustrated in fig. 3.

Table II. gives the results of experiments made on the sets marked 2, 3, 4, 5, 6 by means of the secohmmeter; the set marked 6 being tested when each pair of coils were joined up in parallel so as to increase the inductance as well as when joined up so as to diminish it.

TABLE I.

Particulars of Portable Stand Coils.

Mark on the Stand.	Resistance, in Ohms.	Number of Spirals in the Set.	Length of each Spiral.	Number of Convolutions in each Spiral.		Outside diameter of the Convolutions.		Size of the Wire (S.W.G.)
			inches.			inches.		
1	1·8	6	18¾	78		$1\frac{13}{16}$		No. 12
2	2·54	12	16½	63		$1\frac{3}{16}$		No. 10
3	8·59	12	16½	84		$1\frac{1}{16}$		No. 13
				Outer.	Inner.	Outer.	Inner.	
						inches.	inches.	
4	1·4	20 double, that is 40 coils in all.	16½	15	21	1⅛	⅞	No. 13
5	5·0	" "	16½	27	37	1⅛	⅞	No. 15½
6	99	" "	17	54	79	¾	½	No. 27

TABLE II.

Portable Platinoid Spiral Resistances (fig. 3).

Mark on the Stand.	Winding.	Resistance, in Ohms.	Inductance, in Secohms.	Time-constant, in Seconds.
2 ...	Set of Singly-Wound Spirals ...	2·54	0·000059	23×10^{-6}
3 ...	Set of Single-Wound Spirals ...	8·59	0·00015	18×10^{-6}
6	Set of Doubly-Wound Spirals; Current circulating in all in the *Same* Direction	98·1	0·0021	21×10^{-6}
6	Same set of Doubly-Wound Spirals; Current circulating in *Opposite* Directions.........	99	0·00013	$1·3\times10^{-6}$
4	Set of Doubly-Wound Spirals; Current circulating in *Opposite* Directions	1·4	0·000004	$2·8\times10^{-6}$
5	Set of Doubly-Wound Spirals; Current circulating in *Opposite* Directions	5·0	0·0000094	$1·9\times10^{-6}$

The mean time-constant, therefore, of one of our sets of spirals, whether singly-wound or doubly-wound, provided that the current circulates in all the convolutions in the same direction, is about 21×10^{-6}; whereas if the spirals be doubly-wound and the current be circulating in opposite directions in the two sets of spirals, the mean time-constant is about 2×10^{-6}, or about one tenth of the preceding.

As there happens to be in the laboratory a solenoid, the silk-covered copper wire on which is wound in a series of concentric sections, it was thought interesting to compare the time-constant of two of the sections joined up in parallel, when the current circulated round both in the same direction, with the time-constant when it circulated in opposite directions. Figs. 4 and 5, Plate X., show the direction of the current in the two cases.

	Resistance, in Ohms.	Self-induction, in Sechoms.	Time-constant, in Seconds.
Currents flowing round both coils in the same direction (fig. 4) ...	0·346	0·00059	17×10^{-6}
Currents flowing round the coils in the opposite direction (fig. 5)	0·346	0·000024	$0{\cdot}69 \times 10^{-6}$

The self-induction and time-constant have therefore been reduced to one twenty-fourth part by sending the currents in opposite directions round the two, but the method of winding shown in fig. 5 could not, of course, be employed with the non-inductive resistances constructed of bare wire for large currents described in this communication; since the fundamental condition—*that parts differing much in potential should not be near one another*—would not be fulfilled.

XXX. *A Theory concerning the Constitution of Matter.*

By CHARLES V. BURTON, *D.Sc.*[*]

THE theory described in the following pages is based essentially on one fundamental hypothesis; without the aid of any further assumptions, equations of motion can be deduced, and these, when simplified by certain conditions of symmetry, lead at once to Newton's Laws. In dealing—vaguely enough—with other problems, such as gravitation and the discrete nature of atoms, it is found necessary to make further limitations; and a few suggestions are also made whose nature is purely speculative. But there remains the one central idea, whose development is the especial object of this paper, and in the concluding paragraphs the doctrine will be stated in its most general form and the arguments once more briefly enumerated.

---

[*] Read November 20, 1891.

1. Space, so far as we know it, is filled with a medium, whose ultimate nature may be fluid, but which, owing to turbulent motion or some other cause, has elastic properties resembling in some respects those of a solid. The resistance offered by such a medium to the motion of material bodies presents a problem of some difficulty, *so long as we suppose an atom to consist always of the same portion of æthereal or other substance*; but further on it will be shown that, with a different assumption, the question may assume a new aspect. Before leaving this subject, however, I may point out what appears to me a very serious difficulty of the present view: a perfect vacuum is (at least very nearly) a perfect insulator, and air also insulates well, so that the æther surrounding an electrically charged terrestrial body may remain in a state of stress for a considerable period without appreciable progressive yielding, while at the same time this charged body is being whirled through the æther, which continually gives way like an almost perfect liquid.

2. *Fundamental Assumption. Strain-Figure.*—Consider a region, either infinite or having very distant boundaries, and filled with a homogeneous isotropic elastic medium, whose condition throughout is one of stable equilibrium for small strains of any type. Let the medium now be strained, and held in its strained condition by some compelling agency: there will be a corresponding distribution of stress in the medium, and, provided the strain has at no point too great a value, the original condition will be completely regained after the compelling agency has been removed. But suppose that, instead, the medium is strained further and further from its initial state, and suppose that the restoring stresses do not

always increase with the strain, but that beyond a certain point in the process they begin to fall off in value, until at last a point is reached at which the general tendency of the stress is to further increase the strain. If the compelling agency is now withdrawn, the medium will subside into a new condition of stable equilibrium, involving stress and strain at every point. The state of things thus impressed on the medium is, according to my view, an atom or a constituent of an atom ; it will hereafter be referred to as a "*strain-figure*," and we may now proceed to examine its dynamical properties.

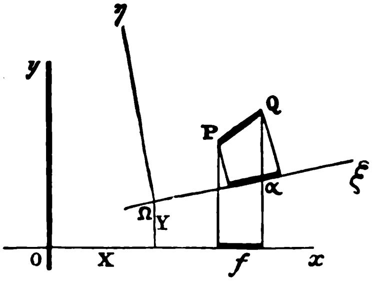
3. *Rigid Body Displacements.*—A strain-figure, being of itself in equilibrium, will remain in equilibrium if transferred to some other portion of the medium, or if its orientation with respect to the medium is changed (for originally the medium was homogeneous and isotropic); we may therefore give to the strain-figure any displacement of which a rigid body would be capable, and the resulting condition of the medium will be one of equilibrium ; there is no statical resistance to such displacement, and no question of the medium giving way.

4. *Equations of Motion.*—If a strain-figure is in motion through the medium, certain conditions must be satisfied in order that its degrees of freedom may not be more than those of a rigid body ; in order, that is, that the strain-figure may retain the same form as if it were at rest in the medium. For suppose that disturbances of the same type as the strains in the strain-figure are propagated through the medium with velocity V; then obviously a necessary condition is that the translational velocity of the strain-figure must be very small compared with V, the rotational motion being (in general) subject to a similar restriction, which cannot be quite so simply expressed. If V is (as I imagine) the velocity with which gravitation is propagated, it is a quantity whose finiteness has not yet been demonstrated, and compared with which all known molar and molecular motions are extremely slow. Another condition to be fulfilled by the motion of the strain-figure is this : the acceleration must be very small compared with V/T, where VT is the radius of the smallest spherical surface within which the whole appreciable *effective mass* (17 *a*) of the strain-figure may be considered to reside.

Assuming, then, that the necessary conditions are satisfied, and that the strain-figure has only six degrees of freedom, let us choose three rectangular axes (those of ξ, η, ζ, with origin Ω) fixed in the strain-figure, and three other rectangular axes (those of x, y, z, with origin O) fixed in the medium and in space; let the coordinates of Ω referred to Ox, Oy, Oz be (X, Y, Z), and let the direction-cosines of the ξ, η, ζ axes be given by the scheme

$$\begin{array}{c|ccc} & x-X & y-Y & z-Z \\ \hline \xi & l_1 & m_1 & n_1 \\ \eta & l_2 & m_2 & n_2 \\ \zeta & l_3 & m_3 & n_3 \end{array} \quad . \quad . \quad . \quad . \quad (1)$$

(For convenience the diagram is only drawn in two dimensions.)



Let P (ξ, η, ζ) be the *undisturbed* position of a certain volume-element of æther (that is, the position which this element would occupy if the strain-figure were non-existent), and let Q ($\xi+\alpha$, $\eta+\beta$, $\zeta+\gamma$) be the *actual* position of the same element. Then, in accordance with the geometrical conditions of motion,

$$\alpha,\ \beta,\ \gamma \text{ are functions of } \xi,\ \eta,\ \zeta \text{ only}; \quad . \quad . \quad . \quad (2)$$

they are the components of displacement of the element whose undisturbed position would have been ξ, η, ζ.

If the components of P Q parallel to Ox, Oy, Oz be called f, g, h respectively, we shall have, after the manner of (1), the scheme

$$\begin{array}{c|ccc} & f & g & h \\ \hline \alpha & l_1 & m_1 & n_1 \\ \beta & l_2 & m_2 & n_2 \\ \gamma & l_3 & m_3 & n_3 \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the *original* density of the æther be taken as unity; and consider that portion of æther which, in the *undisturbed* condition of the medium, would have filled the volume-element $d\xi\, d\eta\, d\zeta$ with coordinates (ξ, η, ζ). Since f, g, h are the displacements parallel to Ox, Oy, Oz, *of the portion of æther so defined*, $\dot{f}, \dot{g}, \dot{h}$ will be the actual velocity-components of this portion, and the energy due to the motion of the strain-figure is

$$T = \tfrac{1}{2}\iiint(\dot{f}^2 + \dot{g}^2 + \dot{h}^2)\, d\xi\, d\eta\, d\zeta. \quad . \quad . \quad . \quad . \quad (4)$$

Now let the axes $\Omega\xi$, $\Omega\eta$, $\Omega\zeta$ be instantaneously coincident in direction with Ox, Oy, Oz respectively, so that

$$l_1 = m_2 = n_3 = 1\,; \quad l_2 = l_3 = m_3 = m_1 = n_1 = n_2 = 0, \quad . \quad . \quad (5)$$

and

$$\dot{l}_1 = \dot{m}_2 = \dot{n}_3 = 0\,;\ \dot{n}_2 = -\dot{m}_3 = \omega_1\,;\ \dot{l}_3 = -\dot{n}_1 = \omega_2\,;\ \dot{m}_1 = -\dot{l}_2 = \omega_3, \quad (6)$$

where ω_1, ω_2, ω_3 are the angular velocity-components of the strain-figure about the axes of ξ, η, ζ or of x, y, z; relations which, in general, are only instantaneously true.

From (3),

$$\dot{f} = l_1\dot{\alpha} + l_2\dot{\beta} + l_3\dot{\gamma} + \dot{l}_1\alpha + \dot{l}_2\beta + \dot{l}_3\gamma;$$

and writing

$$\frac{\partial\alpha}{\partial\xi} \equiv \alpha_1\,; \quad \frac{\partial\alpha}{\partial\eta} \equiv \alpha_2; \quad \frac{\partial\alpha}{\partial\zeta} \equiv \alpha_3\,; \text{ \&c.,} \quad . \quad . \quad (7)$$

the last equation becomes

$$\dot{f} = l_1(\alpha_1\dot{\xi} + \alpha_2\dot{\eta} + \alpha_3\dot{\zeta}) + l_2(\beta_1\dot{\xi} + \beta_2\dot{\eta} + \beta_3\dot{\zeta}) + l_3(\gamma_1\dot{\xi} + \gamma_2\dot{\eta} + \gamma_3\dot{\zeta})$$
$$+ \dot{l}_1\alpha + \dot{l}_2\beta + \dot{l}_3\gamma. \quad . \quad . \quad . \quad (8)$$

Again, from (1),

$$\xi = l_1(x - X) + m_1(y - Y) + n_1(z - Z),$$

so that

$$\dot{\xi} = \dot{l}_1(l_1\xi + l_2\eta + l_3\zeta) + \dot{m}_1(m_1\xi + m_2\eta + m_3\zeta) + \dot{n}_1(n_1\xi + n_2\eta + n_3\zeta)$$
$$- l_1\dot{X} - m_1\dot{Y} - n_1\dot{Z}\,;$$

or, using (5) and (6),

Similarly

and

$$\left.\begin{aligned} \dot{\xi} &= \omega_3\eta - \omega_2\zeta - \dot{X}. \\ \dot{\eta} &= \omega_1\zeta - \omega_3\xi - \dot{Y}, \\ \dot{\zeta} &= \omega_2\xi - \omega_1\eta - \dot{Z} \end{aligned}\right\}$$

Substituting these values in (8), and again using (5) and (6), we obtain

$$\left.\begin{aligned}
&\dot f = -\alpha_1\dot{\mathrm{X}} - \alpha_2\dot{\mathrm{Y}} - \alpha_3\dot{\mathrm{Z}} + \omega_1(\alpha_2\zeta - \alpha_3\eta) + \omega_2(\alpha_3\xi - \alpha_1\zeta + \gamma) \\
&\text{Similarly} \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \omega_3(\alpha_1\eta - \alpha_2\xi - \beta). \\
&\dot g = -\beta_1\dot{\mathrm{X}} - \beta_2\dot{\mathrm{Y}} - \beta_3\dot{\mathrm{Z}} + \omega_1(\beta_2\zeta - \beta_3\eta - \gamma) + \omega_2(\beta_3\xi - \beta_1\zeta) \\
&\text{and} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \omega_3(\beta_1\eta - \beta_2\xi + \alpha), \\
&\dot h = -\gamma_1\dot{\mathrm{X}} - \gamma_2\dot{\mathrm{Y}} - \gamma_3\dot{\mathrm{Z}} + \omega_1(\gamma_2\zeta - \gamma_3\eta + \beta) + \omega_2(\gamma_3\xi - \gamma_1\zeta - \alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \omega_3(\gamma_1\eta - \gamma_2\xi).
\end{aligned}\right\} \quad (9)$$

If these values are substituted in (4), we obtain immediately a complete expression for the energy due to the motion of the strain-figure; and this expression involves only $\dot{\mathrm{X}}$, $\dot{\mathrm{Y}}$, $\dot{\mathrm{Z}}$, ω_1, ω_2, ω_3, and quantities which remain constant throughout the motion. We may therefore apply the principle of moving axes to find the components F, G, H of effective force on the strain-figure; thus

$$\mathrm{F} = \frac{d}{dt}\,\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{X}}} - \omega_3\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{Y}}} + \omega_2\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{Z}}},$$

which by means of (4) and (9) becomes

$$\begin{aligned}
\mathrm{F} = {}&(11)\ddot{\mathrm{X}} + (12)\ddot{\mathrm{Y}} + (31)\ddot{\mathrm{Z}} \\
&- (12)\omega_3\dot{\mathrm{X}} - (22)\omega_3\dot{\mathrm{Y}} - (23)\omega_3\dot{\mathrm{Z}} \\
&+ (31)\omega_2\dot{\mathrm{X}} + (23)\omega_2\dot{\mathrm{Y}} + (33)\omega_2\dot{\mathrm{Z}} \\
&+ \{(31\eta) - (12\zeta) + (\beta 1\gamma)\}\dot\omega_1 + \{(11\zeta) - (31\xi) + (\gamma 1\alpha)\}\dot\omega_2 \\
&\qquad\qquad\qquad\qquad + \{(12\xi) - (11\eta) + (\alpha 1\beta)\}\dot\omega_3 \\
&+ \{(31\zeta) - (33\xi) + (\gamma 3\alpha)\}\omega_2^2 + \{(12\eta) - (22\xi) - (\alpha 2\beta)\}\omega_3^2 \\
&+ \{2(23\xi) - (12\zeta) - (31\eta) + (\alpha 3\beta) - (\gamma 2\alpha)\}\omega_2\omega_3 \\
&+ \{(22\zeta) - (23\eta) + (\beta 2\gamma)\}\omega_3\omega_1 + \{(33\eta) - (23\zeta) + (\beta 3\gamma)\}\omega_1\omega_2,
\end{aligned} \quad (10)$$

with similar values for G and H, where we write

$$\left.\begin{aligned}
&\iiint(\alpha_1^2 + \beta_1^2 + \gamma_1^2)\,d\xi\,d\eta\,d\zeta \equiv (11), \text{ \&c.,} \\
&\iiint(\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3)\,d\xi\,d\eta\,d\zeta \equiv (23) \equiv (32), \text{ \&c.,} \\
&\iiint(\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3)\xi\,d\xi\,d\eta\,d\zeta \equiv (23\xi) \equiv (32\xi), \text{ \&c.,} \\
&\iiint(\beta_1\gamma - \gamma_1\beta)\,d\xi\,d\eta\,d\zeta \equiv (\beta 1\gamma) = -(\gamma 1\beta), \text{ \&c.}
\end{aligned}\right\} \quad (11)$$

Again, if L, M, N are the effective couples,

$$
\begin{aligned}
\mathrm{L} = {} & \frac{d}{dt}\cdot\frac{\partial \mathrm{T}}{\partial \omega_1} - \dot{\mathrm{Z}}\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{Y}}} + \dot{\mathrm{Y}}\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{X}}} - \omega_3\frac{\partial \mathrm{T}}{\partial \omega_2} + \omega_2\frac{\partial \mathrm{T}}{\partial \omega_3} \\
= {} & \{(12\zeta)-(31\eta)-(\beta 1\gamma)\}\ddot{\mathrm{X}} + \{(23\eta)-(22\zeta)+(\beta 2\gamma)\}\ddot{\mathrm{Y}} \\
& \qquad + \{(33\eta)-(23\zeta)+(\beta 3\gamma)\}\ddot{\mathrm{Z}} \\
& + (23)\dot{\mathrm{Y}}^2 - (23)\dot{\mathrm{Z}}^2 + \{(33)-(22)\}\dot{\mathrm{Y}}\dot{\mathrm{Z}} - (12)\dot{\mathrm{Z}}\dot{\mathrm{X}} + (31)\dot{\mathrm{X}}\dot{\mathrm{Y}} \\
& + \{(12\xi)-(11\eta)+(\alpha 1\beta)\}\omega_2\dot{\mathrm{X}} + \{(31\xi)-(11\zeta)-(\gamma 1\alpha)\}\omega_3\dot{\mathrm{X}} \\
& + \{(33\eta)-(23\zeta)+(\beta 3\gamma)\}\omega_1\dot{\mathrm{Y}} + \{(22\zeta)-(23\eta)-(\beta 2\gamma)\}\omega_1\dot{\mathrm{Z}} \\
& + \{(22\xi)-(33\xi)+(31\zeta)-(12\eta)+(\gamma 3\alpha)+(\alpha 2\beta)\}(\omega_2\dot{\mathrm{Y}}-\omega_3\dot{\mathrm{Z}}) \\
& + \{2(23\xi)-(12\zeta)-(31\eta)+(\alpha 3\beta)-(\gamma 2\alpha)\}(\omega_3\dot{\mathrm{Y}}+\omega_2\dot{\mathrm{Z}}) \\
& + \{(22\zeta^2)+(33\eta^2)-2(23\eta\zeta)-(\beta 2\gamma\zeta)+(\beta 3\gamma\eta)\}\dot{\omega}_1 \\
& + \{(23\zeta\xi)+(31\eta\zeta)-(33\xi\eta)-(12\zeta^2)-(\gamma 2\alpha\zeta)+(\gamma 3\alpha\eta) \\
& \qquad -(\beta 3\gamma\xi)+(\beta 1\gamma\zeta)\}\dot{\omega}_2 \\
& + \{(12\eta\zeta)+(23\xi\eta)-(22\zeta\xi)-(31\eta^2)-(\alpha 2\beta\zeta)+(\alpha 3\beta\eta) \\
& \qquad -(\beta 1\gamma\eta)+(\beta 2\gamma\xi)\}\dot{\omega}_3 \\
& + \{(\alpha 3\beta\xi)-(\alpha 1\beta\zeta)+(\gamma 1\alpha\eta)-(\gamma 2\alpha\xi)+(\beta\gamma)\}(\omega_3^2-\omega_2^2) \\
& + \{\alpha 2\beta\xi)-(\alpha 1\beta\eta)+(\gamma 3\alpha\xi)-(\gamma 1\alpha\zeta)+(\beta^2)-(\gamma^2)\}\omega_2\omega_3 \\
& + \{(33\xi\eta)+(12\zeta^2)-(23\zeta\xi)-(31\eta\zeta)+(\gamma 2\alpha\zeta)-(\gamma 3\alpha\eta) \\
& \qquad +(\beta 3\gamma\xi)-(\beta 1\gamma\zeta)\}\omega_3\omega_1 \\
& + \{(12\eta\zeta)+(23\xi\eta)-(22\zeta\xi)-(31\eta^2)-(\alpha 2\beta\zeta)+(\alpha 3\beta\eta) \\
& \qquad -(\beta 1\gamma\eta)+(\beta 2\gamma\xi)\}\omega_1\omega_2, \quad . \quad . \quad (12)
\end{aligned}
$$

with similar values for M and N, where, in addition to (11), we have

$$
\left.\begin{aligned}
&\iiint(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)\eta\zeta\, d\xi\, d\eta\, d\zeta \equiv (12\eta\zeta) \text{ \&c.}, \\
&\iiint(\beta_1\gamma-\gamma_1\beta)\zeta\, d\xi\, d\eta\, d\zeta \equiv (\beta 1\gamma\zeta) \text{ \&c.}, \\
&\iiint\beta\gamma\, d\xi\, d\eta\, d\zeta \equiv (\beta\gamma) \text{ \&c.}
\end{aligned}\right\} \quad (13)
$$

Mr. G. H. Bryan has pointed out to me that the equations of motion (10) and (12) are of exactly the *same form* as those of a solid immersed in a perfect liquid.

5. If U is the force-function of the impressed forces, we shall have of course

$$
\mathrm{F} = \frac{\partial \mathrm{U}}{\partial \mathrm{X}}, \quad \mathrm{G} = \frac{\partial \mathrm{U}}{\partial \mathrm{Y}}, \quad \mathrm{H} = \frac{\partial \mathrm{U}}{\partial \mathrm{Z}},
$$

and evidently

$$\frac{\partial U}{\partial X}=\iiint\left(a\frac{\partial f}{\partial X}+b\frac{\partial g}{\partial X}+c\frac{\partial h}{\partial X}\right)d\xi'\,d\eta'\,d\zeta'\,;$$

where $d\xi'\,d\eta'\,d\zeta'$ is a volume-element in the *actual* condition of the medium, f, g, h the displacements of the æther which now fills this volume-element, and $a\,d\xi'\,d\eta'\,d\zeta'$ &c., the forces exerted on the element of æther. Remembering (1), (3), (5), and (7), we may write the last equation

$$\frac{\partial U}{\partial X}=-\iiint(a\alpha_1+b\beta_1+c\gamma_1)\,d\xi'\,d\eta'\,d\zeta'\,; \quad . \quad . \quad (14)$$

$\dfrac{\partial U}{\partial Y}$ and $\dfrac{\partial U}{\partial Z}$ having similar values, and $\dfrac{\partial U}{\partial \theta_1}$ &c. being similarly obtainable.

We may also express a, b, c in terms of the components of stress $[\xi\xi]$, $[\eta\xi]$ &c.; thus

$$\left.\begin{aligned} a&=-\frac{\partial}{\partial\xi}[\xi\xi]-\frac{\partial}{\partial\eta}[\eta\xi]-\frac{\partial}{\partial\zeta}[\zeta\xi],\\ b&=-\frac{\partial}{\partial\xi}[\xi\eta]-\frac{\partial}{\partial\eta}[\eta\eta]-\frac{\partial}{\partial\zeta}[\zeta\eta],\\ c&=-\frac{\partial}{\partial\xi}[\xi\zeta]-\frac{\partial}{\partial\eta}[\eta\zeta]-\frac{\partial}{\partial\zeta}[\zeta\zeta]. \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

In this section it has been virtually assumed that the strain-figure is exactly superimposed on the otherwise existing condition of the medium.

6. If the motion of the strain-figure is one of pure translation, ω_1, ω_2, ω_3 are all constantly zero, and equation (10) reduces to

$$\text{at the same time}\quad \left.\begin{aligned} F&=(11)\ddot{X}+(12)\ddot{Y}+(31)\ddot{Z}\,;\\ G&=(12)\ddot{X}+(22)\ddot{Y}+(23)\ddot{Z}\,;\\ H&=(31)\ddot{X}+(23)\ddot{Y}+(33)\ddot{Z}\,; \end{aligned}\right\} \quad . \quad . \quad (16)$$

Now construct the ellipsoid

$$(11)\xi^2+(22)\eta^2+(33)\zeta^2+2(23)\eta\zeta+2(31)\zeta\xi+2(12)\xi\eta=M\epsilon^2; \quad (17)$$

and it is evident from (16) above that when the motion is one of pure translation, the effective force (components F, G, H) is perpendicular to the diametral plane of the acceleration (components $\ddot{X}$, $\ddot{Y}$, $\ddot{Z}$), taken with respect to the ellipsoid (17).

(11), (22), (33), which are the sums of squares, are the values of the effective mass in the directions of the axes of reference; while (23), (31), (12), which are the sums of products, correspond to products of inertia in Rigid Dynamics, and may be called "products of mass." It may easily be shown that if r is the length of any given radius of the ellipsoid (17), the effective mass reckoned in the direction of this radius is $M\epsilon^2/r^2$. It may also be remarked that even when the motion is one of pure translation, there will in general be finite effective couples, as is immediately evident from (12).

7. *Case of Symmetry.*—Consider now the particular case in which the strain-figure is symmetrical about a point, and let this point be Ω, the origin of the axes of ξ, η, ζ. We shall then have

$$\left.\begin{array}{l}(11)=(22)=(33)=\text{effective mass};\\(23)=(31)=(12)=0.\end{array}\right\}\quad . \quad . \quad (17a)$$

It is also evident that a rotation of the strain-figure about any axis through its centre of symmetry corresponds to no physical change whatever; there is no possibility of such rotation, nor can any influence exerted on the medium have a tendency to turn the strain-figure about its centre of symmetry. Hence *a strain-figure symmetrical about a point is dynamically equivalent to a single particle of mass*

$$\iiint(\alpha_1^2+\beta_1^2+\gamma_1^2)d\xi\,d\eta\,d\zeta=\iiint(\alpha_2^2+\beta_2^2+\gamma_2^2)d\xi\,d\eta\,d\zeta$$
$$=\iiint(\alpha_3^2+\beta_3^2+\gamma_3^2)d\xi\,d\eta\,d\zeta$$

placed at its centre of symmetry.

It is worth remarking that the theory now considered, though differing so widely from the theory of Boscovich, leads to the same dynamical results.

8. So far we have only considered the case of a single

strain-figure. When there are more than one, the experimental properties of matter lead us to suppose :—

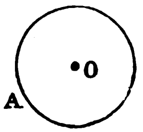
(i) That the entire distribution of displacement in the medium is to be found (at least very approximately) by compounding geometrically the distributions which the various strain-figures would have produced separately.

(ii) That the strain-figures exert forces upon one another, thus changing or tending to change their motion through the medium.

In speaking of the medium I have more particularly in mind the hypothesis of a turbulently moving liquid, as propounded by Sir W. Thomson * and modified by Prof. Fitzgerald †. Without entering on the subject mathematically (which I am unable to do), it may be noticed that, according to this view, the æther is made up of interlacing vortex-filaments, the interspaces between which are filled with quiescent or irrotationally moving liquid. The network of filaments is equivalent to a sponge-like compressible solid, whose pores, however, are completely filled with incompressible fluid. The medium, *taken as a whole*, is equivalent to an incompressible solid ; but as the effectively liquid (*i. e.* irrotationally moving) portion of the medium cannot suffer any strain which calls up an opposing stress, it appears that in considering the strain-figure we should have to deal with the sponge-like compressible solid, made up of vortex-filaments.

9. *Gravitation.*—Disregarding this last speculation, let us return to the more general question of strain, and for simplicity take the case when the strain-figure is symmetrical about a point O. So far as this one strain-figure is concerned, we may, roughly speaking, divide the medium into two portions, in one of which the strain tends to increase, while in the other it tends to decrease. If these portions are separated by a single surface, then, in the present case of symmetry, this surface (A) will be spherical. On one side of A the medium will be so strained that a small additional strain of the same type would correspond to increased potential

Fig. 2.



---

* B. A. Report, 1887, p. 486; Phil. Mag., October 1887, p. 342.

† 'Nature,' May 9, 1889.

energy, while on the other side of A a small increase of strain would correspond to decreased potential energy. If the strain *outside* A corresponds to *increased* potential energy, two distant strain-figures will *repel* one another ; but if the strain *outside* A corresponds to *decreased* potential energy, two distant strain-figures will *attract* one another. If gravitation is to be explained in accordance with the theory of this paper, the above would seem to indicate the nature of the explanation. There must be some type of strain which would of itself be produced in the medium, were it not that in some other portion of the medium there would be an accompanying strain of a type corresponding to stability. Perhaps we may also vaguely infer why the gravitative attraction exerted by a body is proportional to its mass, for we have supposed both these quantities to be determined by the distribution of strain in the medium ; it seems possible, indeed, that the greater part of the effective mass of a strain-figure might reside in that region where attraction follows the Newtonian law.

10. *Collision.*—If we bring the centres of two strain-figures close together, so that the spheres A intersect sufficiently far, we shall be partly superposing two strain-distributions of such a type that the energy increases with the strain, so that repulsion may, perhaps, ensue. Of course this is only a very rough attempt at explaining the effects of collision, and takes no account of any *deformation* which might be produced in the strain-figures on bringing their centres to such close quarters. But even were such deformation produced, the results of a collision might still be of a very simple character. So long as the condition of § 4 is satisfied, *so long, that is, as the motions concerned are extremely slow compared with* V (*the velocity with which gravitation is propagated*) *the distribution of displacement in the medium will be at each instant the same as if the two strain-figures were at rest. Hence, though the two strain-figures may recoil from one another with altered velocities they will not have acquired any vibratory motion from the impact.* This statement is equally true whether the strain-figure is symmetrical or not, but in the case of symmetry about a point, the properties of a strain-figure will be identical with those of a Boscovich particle, exerting actions at a distance according to a law of force.

11. The result just established has a bearing on the dynamical theory of heat. We know that the number of degrees of freedom of a molecule, and consequently also of an atom, is probably finite. If an atom consisted of n strain-figures of the most general kind, the total number of degrees of freedom would be $6n$; but if each of the constituent strain-figures were symmetrical about a point the number would be reduced to $3n$. Since the spectra of elementary vapours teach us that the number of degrees of freedom, though finite, is usually very large, we are led to infer that most atoms are formed by the aggregation of a large number of strain-figures.

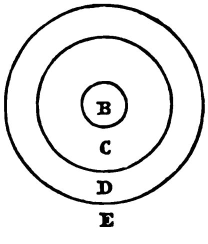
12. *Discrete Nature of Atoms.*—Why do the atoms form a discrete series, and why are all atoms of a given element identical in physical and chemical properties? These are questions which it would be difficult to answer, though it does not seem quite so difficult to point out a direction in which we may possibly look for an explanation. If an atom consists of a number of points endowed with inertia and with the power of exerting actions at a distance, it is hard to see how any explanation can be given: nor is the difficulty much less if the atom consists of a vortex ring or a vortex tangle, for we should then simply have to infer that the existing rings or tangles are as they are because they have always been so, and that an indefinite variety of rings or tangles of intermediate forms and sizes would be *à priori* perfectly possible. But if the atom consists of one or more strain-figures, the question becomes: Why do the strain-figures form a discrete series? Now we have conceived a strain-figure to be a disturbed condition of the medium, *which is of itself in stable equilibrium throughout*, and this immediately imposes an immense restriction on the possible varieties; since, also, the strains are impressed on a medium which would otherwise be homogeneous and isotropic, the conditions essential to stable equilibrium will be the same for all strain-figures, provided, that is, that the proximity of the centres of two or more does not disturb their form. But at this point a further assumption will be necessary; for if the turbulent motion or other structure of the medium were absolutely homogeneous (which implies infinite fine-grainedness), and if a strain-figure defined by a distribution of displacement

A were a possible one, then the figure defined by the distribution A magnified n diameters would be equally possible. Thus the possible strain-figures, though infinitely restricted in variety by the conditions of equilibrium, would form not a discrete but a continuous series, or possibly a discrete system of continuous series. We must suppose, then, that the *coarse-grainedness* of the medium has an influence in determining the *size* of possible strain-figures.

13. *Physical Illustration.*—A very imperfect illustration of this last point may be drawn from a physical phenomenon; for consider a region free from the action of gravity and filled with a saturated vapour, which by some means is maintained at constant temperature and pressure. If compression takes place, liquid will be formed, and will exist in equilibrium with the vapour, no intermediate condition of the substance being consistent with equilibrium and stability. But, neglecting surface-tension, the liquid need not be formed in masses of any special size; that is, the possible sizes of the drops of liquid will form a continuous series. If, however, surface-tension is taken into account, the case is different; for suppose that the vapour is *slightly supersaturated*, and contains a number of spherical drops of liquid. Very large drops will continue to increase, and very small drops will diminish, while drops of one particular size (if any such are present) will just be in equilibrium with the vapour, although the equilibrium is necessarily unstable. But notwithstanding that the analogy fails in this and in many other respects, it may serve to roughly illustrate the suggestion that the size of strain-figures, and consequently of atoms, is determined by the coarse-grainedness of the medium.

14. *Formation of Atoms.*—As regards the different varieties of atoms, we may conceive them to be made up of one kind or of several kinds of strain-figures. In place of the simpler form of fig. 2 we might for example imagine a form like fig. 3, where in the regions C, E the strain is such that the potential energy would decrease with increase of strain, while in the

Fig. 3.



regions B, D the strain and the potential energy would increase together. We might also imagine distributions which were not spherical, in which case it might happen (§ 4) that the laws of motion were less simple than those of Newton. It would not then necessarily follow that an atom consisting of more than one strain-figure would possess the same complex dynamical properties, though I am not aware of any evidence that the separate atoms or molecules of a substance move in accordance with Newton's laws.

Of course the operation described in § 2 is not intended to represent the *formation* of a strain-figure, but merely to show that the *existence* of such a distribution is conceivably possible. If the ultimate fluid had long ago possessed motion of the most general kind, we might imagine its present condition to be due to the degeneration of that motion into a fine-grained turbulence; and if, in the quasi-solid so constituted, the existence of strain-figures were possible, it seems not unlikely that such would incidentally have been formed, unless the motion fulfilled special conditions. I would suggest then, very tentatively, *that if the distribution of motion in the ultimate fluid had fulfilled certain special conditions, there would have been no atomic matter in the universe, and that the existence of matter as we know it is an indication that such conditions were wanting.*

Concerning the possibility of the "transmutation" of elements, this investigation leads to no immediate conclusion, but any conceivable superposition of two strain-figures would probably involve only finite potential energy, so that the effect of a very severe direct encounter might be to make the two strain-figures pass *through* one another. It seems possible, too, that at some stage of the impact the distribution might resolve itself into one or more strain-figures of a different kind, the entire effective mass not being *necessarily* the same as before.

15. It may be remarked that, according to the assumptions of this paper, every motion in the universe is ultimately due to *stationary motion* in a quasi-solid medium. This medium, which extends through all known space, is supposed to be (at least appreciably) perfect in its elastic properties, that is, free from viscous yielding and internal friction; and its ever

changing distribution of stress and strain is to be held accountable for all observed phenomena.

16. The subject of this paper being now explained, so far as the vagueness of my own views will allow, the fundamental proposition may be stated as follows :—

A given portion of matter consists, not of any individual portion of æthereal or other substance, but of modifications in the structure or energy or other qualities of the æther, and when matter moves it is merely these modifications of structure or of energy or of other qualities which are transferred from one portion of the æther to another.

The strain-figure has here been almost exclusively considered, and we have seen

(i) That provided the motion is slow compared with a certain velocity V (§ 4), a strain-figure will encounter no resistance in travelling through the æther, and will obey laws of motion which include Newton's Laws as a particular case.

(ii) That gravitative and inter-atomic forces may possibly be supposed to arise from the stresses which accompany the distribution of strain.

(iii) That a collision between two single strain-figures would not set them vibrating, so that an atom consisting of strain-figures would have a finite number of degrees of freedom, as required by the dynamical theory of heat.

(iv) That the size and nature of possible strain-figures, and therefore also of possible atoms, would be limited by the conditions of equilibrium, thus giving rise, perhaps, to a discrete series.

On the other hand, it is a special difficulty of my theory that we require some assumption as to the superposition of strain-figures to account for the fact that the mass of a material body is equal to the sum of the masses of its constituent particles.

In conclusion, I have to thank Mr. G. H. Bryan for his kindness in verifying some portions of the analysis, as well as for a suggestion which has immensely lessened the labour of calculation.

Note added November 21.

In the discussion which followed this paper, Professor

Fitzgerald mentioned that in his lectures he had spoken of matter as possibly travelling through space in the same way that a drop of water travels through a block of ice. The idea involved is the same as that which forms the basis of the foregoing pages. I have availed myself of Professor Fitzgerald's criticism to amend some remarks contained in the earlier part of the paper, and there is just one thing more which I should like to add. Every opinion expressed in the paper is not to be taken as an essential part of the theory, and, indeed, the most I can hope from an investigation so obviously incomplete is that it may prove suggestive to those who are working at the subject.

XXXI. *Struts and Tie-Rods with Lateral Loads.*

By Professor John Perry, *D.Sc.*, *F.R.S.*[*]

I THINK that this subject has not yet been taken up scientifically; yet it is very important. The practical treatment of the whole subject of struts is in a very unsatisfactory condition; and it is mainly due to this that, of two bridges designed for the same spans and loads, by two engineers, one has sometimes more than twice the weight of the other; and in all probability the one of least weight is in some parts very much too strong, and in other parts has very little strength in excess of what is absolutely necessary.

It will be in the recollection of some of the members present that Professor Ayrton and I, in 1886, showed why experiment always gave a breaking-load for a strut which was less than that which results from Euler's theory.

A strut is a prismatic body of homogeneous material subjected to equal and opposite crushing forces at its ends. Taking its length, $2l$; the least moment of inertia of its cross section about a straight line through its centre of area, I; Young's modulus of elasticity, E; f_c the compressive stress which the material will stand: then for a strut hinged at its ends (that is, if the resultant force at each end acts at the centre of the end), Euler's theory gives the breaking-load as the lesser of the two values

---

[*] Read December 4, 1891.

$$U = \frac{EI\pi^2}{4l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$f_c A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The length of strut for which these two answers are the same is

$$l = \frac{\pi k}{2}\sqrt{\frac{E}{f_c}},$$

if k is the least radius of gyration of the section about a line in the section through its centre of gravity.

Now we showed that want of homogeneity, inaccuracy of loading, and initial want of straightness in the strut all lead to a smaller load being able to break the strut, and, indeed, that an initial want of straightness (the assumption of an initial deflexion α at the middle) may be taken to represent all three kinds of discrepance. We showed that for such values of l as make the two answers (1) and (2) nearly the same, a very small error α produces very great discrepance—a very great diminution in the real breaking-load from Euler's breaking-load. We showed that when a strut is perfect and perfectly loaded, there is no deflexion until the breaking-load is reached; and that if a deflexion be artificially produced, the strut will straighten itself again, whereas in the imperfect strut there is a deflexion which increases at first in proportion to the load, and then more rapidly.

Such deflexions had been observed and carefully tabulated and published by Mr. Hodgkinson and Mr. Edwin Clarke, but hitherto no use had been made of them, and they had been regarded as unaccountable.

We pointed out that the error α in carefully made and carefully loaded struts seems roughly to follow a rule, and consequently that without making the unwarrantable assumption which has hitherto been used to make Gordon's formula appear to be a rational one, we had a right to look upon Gordon's formula as an empirical formula, which, being of the value (2) for short struts and of the value (1) for long struts, might be taken as right for struts of any length.

Since the publication of our paper ('The Engineer,' Dec. 10th, 1886) I have been in the habit of giving to students problems on struts laterally loaded : as when a strut lies horizontally and is loaded with its own weight; or as when a strut like the coupling-rod of a locomotive, or any connecting-rod, is loaded by its own centrifugal force.

In this paper I shall first give the general solution of such problems ; and it will include the old problem of want of straightness of the strut and inaccuracy of loading.

I take as the axis of x the straight line through the centres of area of the two ends of a strut. The origin is at the point of bisection of this line. The length of strut is $2l$. Systems of forces act at the ends such that their resultants are two equal and opposite pushing forces F (or pulling forces F_1) in the axis of x together with certain couples. Thus if a strut is hinged at the ends, and if the resultant forces do not act exactly through the centres of area, but at points whose distances are h (measured parallel to the plane in which bending is most likely to occur) from the centre of area, I take it that there are couples at the ends, of the amounts Fh or $-F_1h$. Again, if a strut is fixed at an end, I take it that there is an end couple $-M_0$ or $-M_1$ whose amount will be determined by given conditions of lateral loading and end push or pull. In this case M_0 and M_1 will include such terms as Fh. I can assume that the strut when unloaded is not quite prismatic, the ordinate of its centre line being y'. It is easy to make the treatment even more general by taking the initial shape a tortuous curve, and by considering bending as taking place in various planes ; but for nearly all practical purposes it is sufficient to deal with one plane only.

I suppose the strut to have a lateral loading such that if it were a beam *supported* merely at the ends, the bending-moment would be $\phi(x)$[*].

It can readily be shown that the total bending-moment at

---

[*] If the strut is loaded uniformly laterally with a total load W,

$$\phi(x)=\tfrac{1}{4}W(l^2-x^2).$$

If the strut has a single load at the middle there is a discontinuity in the function at the middle; but we may put it

$$\phi(x)=\tfrac{1}{2}W(l-\sqrt{x^2}),$$

the minus sign being taken whether x is positive or negative. There are easy graphical methods, well known to all practical engineers, for finding $\phi(x)$ for any system of lateral loading however complicated.

any section of a strut or tie-rod, if $-\mathrm{M}_0$ and $-\mathrm{M}_1$ are the couples at its ends, is

$$\mathrm{F}y+\phi(x)-\tfrac{1}{2}(\mathrm{M}_0+\mathrm{M}_1)+x\frac{\mathrm{M}_0-\mathrm{M}_1}{2l}, \quad . \quad . \quad . \quad (3)$$

or

$$-\mathrm{F}_1y+\phi(x)-\tfrac{1}{2}(\mathrm{M}_0+\mathrm{M}_1)+x\frac{\mathrm{M}_0-\mathrm{M}_1}{2l}. \quad . \quad . \quad . \quad (4)$$

The curvature being always small is $\pm\frac{d^2y}{dx^2}$, and we take the $+$ or $-$ sign according to the sign given to bending-moment, and hence we have generally for struts

$$\mathrm{F}y+\phi(x)-\tfrac{1}{2}(\mathrm{M}_0+\mathrm{M}_1)+x\frac{\mathrm{M}_0-\mathrm{M}_1}{2l}=-\mathrm{EI}\left(\frac{d^2y}{dx^2}-\frac{d^2y'}{dx^2}\right),$$

or

$$\frac{d^2y}{dx^2}+\frac{\mathrm{F}}{\mathrm{EI}}y=\frac{d^2y'}{dx^2}-\frac{1}{\mathrm{EI}}\phi(x)+\frac{1}{2\mathrm{EI}}(\mathrm{M}_0+\mathrm{M}_1)-\frac{x}{\mathrm{EI}}\frac{\mathrm{M}_0-\mathrm{M}_1}{2l};(5)$$

and for tie-rods we have merely to write $-\mathrm{F}_1$ instead of F.

I shall assume for the present that EI is constant everywhere.

Now $\phi(x)$ can always be developed in a Fourier's series. But it will sometimes be found convenient to express it in the form

$$\phi(x)=a_0+rx+tx^2+a_1\cos sx+b_1\sin sx+$$
$$a_2\cos 2sx+b_2\sin 2sx+\&\mathrm{c.}+a_i\cos isx+b_i\sin isx+\&\mathrm{c.},$$

where $s=\frac{\pi}{2l}$.

Terms in x^3, x^4, &c., may easily be taken also.

We may put this symbolically as

$$\phi(x)=a_0+rx+tx^2+(a_i,\ b_i,\ is). \quad . \quad . \quad . \quad (6)$$

Whatever y' may be, we may express it symbolically as

$$y'=\alpha_0+(\alpha_i,\ \beta_i,\ is). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence (5) may be written:—

For struts:

$$\frac{d^2y}{dx^2}+\frac{\mathrm{F}}{\mathrm{EI}}y=-\frac{a_0}{\mathrm{EI}}+\frac{\mathrm{M}_0+\mathrm{M}_1}{2\mathrm{EI}}-\frac{x}{\mathrm{EI}}\left(r+\frac{\mathrm{M}_0-\mathrm{M}_1}{2l}\right)$$
$$-\frac{t}{\mathrm{EI}}x^2-\left(i^2s^2\alpha_i+\frac{a_i}{\mathrm{EI}},\ i^2s^2\beta_i+\frac{b_i}{\mathrm{EI}},\ is\right); \quad . \quad . \quad (8)$$

and for tie-rods we have the same equation, if $-F_1$ be substituted for F.

Now if n^2 be written instead of $\frac{F}{EI}$ in the case of a strut, and instead of $\frac{F_1}{EI}$ in the case of a tie-rod, we have the following solutions.

For a Strut:—

$$EIy = M\cos nx + N\sin nx + A_0 + Rx + Tx^2 + (A_i, B_i, is), \quad (9)$$

where M and N are arbitrary constants,

$$A_0 = \frac{-a_0 + \frac{1}{2}(M_0 + M_1)}{n^2} + \frac{2t}{n^4}.$$

$$R = -\frac{r}{n^2} + \frac{M_1 - M_0}{2ln^2}.$$

$$T = -\frac{t}{n^2}.$$

$$A_i = -\frac{i^2s^2\alpha_i EI + a_i}{n^2 - i^2s^2}.$$

$$B_i = -\frac{i^2s^2\beta_i EI + b_i}{n^2 - i^2s^2}.$$

For a Tie-rod:—

$$EIy = Me^{nx} + Ne^{-nx} + A_0' + R'x + T'x^2 + (A_i', B_i', is), \quad (10)$$

where

$$n = \sqrt{\frac{F_1}{EI}},$$

where M and N are arbitrary constants,

$$A_0' = \frac{2t}{n^4} + \frac{a_0 - \frac{1}{2}(M_0 + M_1)}{n^2},$$

$$R' = \frac{r}{n^2} - \frac{M_1 - M_0}{2ln^2},$$

$$T' = \frac{t}{n^2},$$

$$A_i' = \frac{a_i + \alpha_i EIi^2s^2}{n^2 + i^2s^2},$$

$$B_i' = \frac{b_i + \beta_i EI i^2 s^2}{n^2 + i^2 s^2}.$$

When y is known, the greatest value of the bending-moment is known. A line through the centre of a section at right angles to the plane of bending is always called the neutral line of the section, even in cases like the present where the name is a little misleading; and if z_c and z_t are the greatest distances of points in the section from the neutral line, on the compressive and on the tensile sides respectively, and if μ is the greatest bending-moment, then in struts, if A is the area of cross section,

$$\frac{\mu}{I} z_c + \frac{F}{A} = f_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{\mu}{I} z_t - \frac{F}{A} = f_t; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and in tie-rods,

$$\frac{\mu}{I} z_c - \frac{F_1}{A} = f_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{\mu}{I} z_t + \frac{F_1}{A} = f_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where f_c and f_t are the maximum compressive and tensile stresses to which any part of the strut or tie-rod is subjected. In many practical cases $z_c = z_t$, and then I/z_c is called Z, the strength-modulus of the section.

We can therefore find the strength and stiffness of any strut of uniform section fixed or free at either or both ends when loaded laterally in any way whatever.

When, as in many practical cases, it is allowable to take an approximate value for $\phi(x)$, and especially when the lateral loading and initial shape are such that the strut is symmetrical about the axis of y, the work can be greatly simplified by not introducing such terms as r and t. In the symmetrical cases such terms as b_i and β_i are absent from the expressions, and $M_0 = M_1$. If the ends are not fixed, $M_0 = M_1 = -Fh$.

Examples.

I. A uniform straight strut has a lateral load W uniformly distributed. It will be found that in this case we have very nearly

$$\phi(x) = \tfrac{1}{4} \mathrm{W} l \cos \frac{\pi}{2l} x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Then, if F is the resultant of the pushing forces, (8) becomes

$$\frac{d^2y}{dx^2} + \frac{\mathrm{F}}{\mathrm{EI}} y = \frac{\mathrm{M}_0}{\mathrm{EI}} - \frac{\frac{1}{4}\mathrm{W}l}{\mathrm{EI}} \cos \frac{\pi}{2l} x. \quad . \quad . \quad (16)$$

Solving this by the rule given above and applying the conditions that $y=0$ when $x=l$, and $\frac{dy}{dx}=0$ when $x=0$, it is easy to see that the solution becomes

$$y = \frac{\mathrm{M}_0}{\mathrm{F}} - \frac{\mathrm{M}_0}{\mathrm{F} \cos l \sqrt{\frac{\mathrm{F}}{\mathrm{EI}}}} \cos x \sqrt{\frac{\mathrm{F}}{\mathrm{EI}}} + \frac{\frac{1}{4}\mathrm{W}l}{\mathrm{EI}\frac{\pi^2}{4l^2} - \mathrm{F}} \cos \frac{\pi}{2l} x. \quad (17)$$

We can at once work out from this, cases in which with such a lateral loading we have any equal couples M_0 applied at the ends, including the case in which the end is hinged, but the resultant push at the ends is applied with the inaccuracy h, in which case $-\mathrm{M}_0 = \mathrm{F}h$. In case the strut is fixed at the ends we apply the condition $\frac{dy}{dx}=0$, when $x=l$. This gives us

$$\mathrm{M}_0 = \frac{\pi \mathrm{W} \sqrt{\mathrm{FEI}}}{8(\mathrm{U}-\mathrm{F})} \cot l \sqrt{\frac{\mathrm{F}}{\mathrm{EI}}}.$$

It is not instructive to pursue this example unless numerical values are taken for the dimensions of the strut and the loading W. If the strut is hinged at the ends and $h=0$, then $\mathrm{M}_0=0$, and

$$y = \frac{\frac{1}{4}\mathrm{W}l}{\mathrm{EI}\frac{\pi^2}{4l^2} - \mathrm{F}} \cos \frac{\pi}{2l} x. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

As this is a simple case, common in practice, I will work out the stresses.

(18) is evidently true even when $F=0$. The deflexion in the middle is

$$y_1=\frac{\frac{1}{4}Wl}{EI\frac{\pi^2}{4l^2}-F}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the greatest bending-moment μ is

$$\mu=Fy_1+\tfrac{1}{4}Wl,$$

or

$$\mu=\tfrac{1}{4}Wl\left(\frac{\frac{EI\pi^2}{4l^2}}{\frac{EI\pi^2}{4l^2}-F}\right). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If $W=0$ and if μ has any value whatever, the denominator must be 0. Putting it equal to 0, we have Euler's law for the strength of struts which are so long that they bend before breaking. If Euler's value of F be called U,

$$U=\frac{EI\pi^2}{4l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Using this as a symbol, (20) becomes

$$\mu=\tfrac{1}{4}Wl\frac{U}{U-F}; \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and we can at once apply (11) and (12) to find the greatest compressive and tensile stresses in the strut. Then, writing $\frac{U}{A}$ as β, and $\frac{F}{A}$ as ω, we see that if f is the compressive stress which the material will bear,

$$\left(1-\frac{\omega}{f}\right)\left(1-\frac{\omega}{\beta}\right)=\frac{Wl}{4fZ}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From this, if we are given the dimensions of the strut and f and W, it is easy to calculate ω. The solution of the quadratic is

$$2\omega=f+\beta-\sqrt{(f+\beta)^2+\left(\frac{Wl}{Z}-4f\right)\beta}, \quad . \quad . \quad (24)$$

the *minus* sign being taken because ω is evidently less than f and less than β.

Coupling-Rods.—Starting from this result, my students have for several years worked out the relative breadths and thicknesses of the sections of coupling-rods and connecting-rods of engines, which are struts whose lateral loads are due mainly to centrifugal force.

Every point in a coupling-rod describes a circle of radius r inches. If the section is, say, rectangular (an elliptic section is just as easy to deal with) of dimensions d in the plane of motion and b at right angles to the plane of motion; taking the whole mass as $2\,l\,b\,d \times {\cdot}28$ divided by 32·2; the centrifugal force

$$W = \frac{lbdrn^2}{62940} \text{ lbs. wt.}$$

if the rod makes n revolutions per minute.

In one direction a coupling-rod is a strut hinged at the ends, and the thrust per square inch that it can receive, assuming that it is properly made, is to be the same as the thrust which it will stand in the other direction. In the first direction it has lateral loading due to centrifugal force.

Now it might be supposed that in the second direction the strut is as if fixed at the ends; but the pins are very short, and the resultant load is certainly not applied axially, and it is safer to assume that in this direction also the strut is as if hinged at the ends. I assume this, although perhaps from the possibility of accidental untrue loading we ought to go further in this safe direction and assume that the strength of the strut is less. Assuming it hinged also in the second direction, however, the thrust per square inch which it can receive is $EI_1 \frac{\pi^2}{4l^2}/bd$, and in this case $I_1 = \frac{db^3}{12}$; so that, if $E = 3 \times 10^7$,

$$\omega = 6{\cdot}17 \frac{b^2}{l^2} \times 10^6.$$

This is the value to use in formula (23).

Taking f for the steel used as 20,000 lb. per square inch, being the proof stress which the material will stand for an infinite number of reversals of stress, we find for equal strength in the two directions, applying (23), that

$$8{\cdot}4 \times 10^8\left(1 - 308\frac{b^2}{l^2}\right)\left(1 - \frac{b^2}{d^2}\right) = n^2 l^2 r / d.$$

Thus, for example, if $b = 1$, and $l = 30$, and $r = 12$, we have the following tables:—

If $b=1$.

d.	n.
1	0
1·5	205
2	277
2·5	327
3	368
4	437
6	545

If $b=1·5$.

d.	n.
1·5	0
2	125
2·5	170
3	202
4	249
6	318
10	440

To illustrate these results. On a certain locomotive engine the coupling-rods are 68 inches long or $l=34''$, $d=3\frac{1}{2}$, $b=1\frac{3}{8}$, $r=12$.

Now it will be found that this rod is equally strong in the two directions if $n=298$ revolutions per minute. The limit of speed which has practically been settled for the engine by the breaking of similar coupling-rods is 258 revolutions per minute.

Without trying to make a rod equally strong to resist bending in the two directions, it is interesting to consider the F or rather ω which a coupling-rod will stand when revolving at n revolutions per minute.

Using (24) and taking

$$E=3\times 10^7,\quad \frac{Wl}{Z}=\frac{l^2rn^2}{10490d},$$

$\beta=6·17\times 10^6\frac{d^2}{l^2}$, so that ω does not depend upon b.

If $l=30$ and $r=12$, and $f=20{,}000$, we find for various values of d and n the following values of ω :—

Values of n.	ω if $d=2$.	ω if $d=3$.	ω if $d=4$.
0	20000	20000	20000
100	16720	18810	19090
200	11760	15455	16930
300	5505	10690	13400
350	2615	7880	11150
394	0	......	......
400	......	5070	8800
450	......	2045	6150
483	......	0	......
500	......	......	3400
557	......	......	0

Taking as before $l=30$ and $r=12$, but now $f=10{,}000$, we find :—

Values of n.	ω if $d=2$.	ω if $d=3$.	ω if $d=4$.
0	10000	10000	10000
100	8166	8995	9500
200	3976	6185	8450
250	1490	4240	......
279	0	......	......
300	......	2010	4200
340	......	0	......
350	......	......	2050
394	.....	......	0

It is quite easy to make similar calculations on sections of **I** shape. Rods of this section may be made equally strong to resist bending in the two directions, at much higher speeds than are possible in the case of the rectangular section. The elliptic section is not so good as the rectangular, and the I section is better than either.

II. A uniform straight tie-rod has a lateral total load W uniformly distributed. The resultant of pulling forces at the ends is F_1 and passes exactly through the centres of the ends ; (8) becomes

$$\frac{d^2y}{dx} - \frac{F_1}{EI}y = +\frac{M_0}{EI} - \frac{\frac{1}{4}Wl}{EI}\cos\frac{\pi}{2l}x. \quad . \quad . \quad (25)$$

Using n^2 for $\frac{F_1}{EI}$,

$$y = Me^{nx} + Ne^{-nx} - \frac{M_0}{n^2EI} + \frac{\frac{1}{4}Wl\cos\frac{\pi}{2l}x}{EI\left(n^2+\frac{\pi^2}{4l^2}\right)}. \quad . \quad (26)$$

Applying the conditions $y=0$ when $x=l$, and $\frac{dy}{dx}=0$ when $x=0$, we find

$$y = \frac{M_0}{EIn^2}\left\{\frac{e^{nx}+e^{-na}}{e^{nl}+e^{-nl}} - 1\right\} + \frac{\frac{1}{4}Wl\cos\frac{\pi}{2l}x}{EI\left(n^2+\frac{\pi^2}{4l^2}\right)}. \quad . \quad . \quad (27)$$

If we suppose that no couples are applied at the ends, or $M_0=0$,

$$y=\frac{\frac{1}{4}Wl\cos\frac{\pi}{2l}x}{F_1+\frac{\pi^2}{4l^2}EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This is identical with (18) if $F_1=-F$. So that in this simple case we can use the same expression for the shape, whether the bar is a strut or a tie-rod.

In the case of a strut,

$$\frac{\frac{1}{4}Wl}{Z}\frac{U}{U-F}+\frac{F}{A}=f_c, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\frac{\frac{1}{4}Wl}{Z}\frac{U}{U-F}-\frac{F}{A}=f_t, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and by taking F in the expressions negative, we find f_c and f_t in a tie-rod.

A numerical example will illustrate these formulæ. I will take a round rod of mild steel lying in a horizontal position, so that it is loaded laterally by its own weight, dimensions being in inches. If d is its diameter, the weight of 1 cubic inch of material is 0·28 lb. and $E=3\times 10^7$,

$$I=\frac{\pi d^4}{64} \text{ and } Z=\frac{\pi d^3}{32}, \quad U=3{\cdot}6\times 10^6\frac{d^4}{l^2},$$

$$f_c \text{ or } f_t=\frac{4{\cdot}03\times 10^6 d^3}{3{\cdot}6\times 10^6\frac{d^4}{l^2}-F}\pm\frac{F}{\frac{\pi}{4}d^2};$$

the *plus* sign being taken in calculating f_c and the *minus* in calculating f_t. Also

$$y_1=\frac{\cdot 11 d^2 l^2}{3{\cdot}6\times 10^6\frac{d^4}{l^2}-F}.$$

Take $d=1$ inch, $l=60$ inches,

$$f_c \text{ or } f_t=\frac{4{\cdot}03\times 10^6}{1000-F}\pm 1{\cdot}273F \text{ and } y_1=\frac{396}{1000-F}.$$

From this I have calculated the following table. The negative values of F indicate that instead of a strut we have a tie.

F.	f_c.	f_t.	y_1.
1000	∞	∞	∞
900	41446	39154	3·96
800	21168	19132	1·98
700	14324	12542	1·32
500	8697	7423	0·792
300	6648	4866	0·566
100	4605	4351	0·440
0	4030	4030	0·396
− 100	3537	3791	0·360
− 300	2718	3482	0·305
− 500	2031	3323	0·264
− 700	1479	3261	0·233
− 900	975	3267	0·208
−1200	304	3360	0·180
−1348	0	3432	0·169

We see, then, that a comparatively small thrust F produces excessive stress in the strut and great deflexion y_1.

We see also that a tensile force F_1 of 900 lb. halves the deflexion of the unpulled beam, and not only reduces the compressive stress to less than a quarter of its old amount, but it has also reduced the tensile stress by nearly 20 per cent. And a tensile load F_1 of 1348 lb. completely destroys all compressive stress in the strut, whilst reducing the greatest tensile stress of the unpulled strut by 15 per cent.

In fact when a beam is very long, pulling out its ends may reduce the compressive stress in it to nothing, whilst also reducing the tensile stress. A longer or smaller beam than that which I have taken will illustrate this matter even better.

The tension F_1 which will make f_t a minimum is

$$F_1=\sqrt{\left(\frac{AWl}{4Z}U\right)}-U,$$

and when $\frac{AWl}{4Z}$ is so small as U then the unpulled beam itself has the minimum tensile stress.

If $\frac{AWl}{4Z}$ is less than U it requires a pushing force F to give the minimum tensile stress.

In all cases there is, of course, a pulling force F_1 which will cause the beam to have no compressive stress in it. In some cases there is a pushing force F which will cause the beam to have no tensile stress in it, but this can only be the case when

$$U \text{ is not less than } \frac{WlA}{Z}.$$

In this case, a beam to carry lateral load may be built up of separate blocks of material which have just sufficient friction to prevent slipping due to shearing forces at the joints. It is a case which may be treated graphically by Professor Fuller's Method, as the change of shape is small.

It will be seen from the above simple example that very instructive numerical problems may be given. Thus if, instead of the lateral loading, we have merely couples, $-M_0$ or m, applied to the ends of a strut, we use $W = 0$ in (17).

When $x=0$ we have the greatest value of y_1, or

$$y_1 = \frac{m}{F}\left\{\frac{1}{\cos l\sqrt{\frac{F}{EI}}} - 1\right\}.$$

Also $Fy_1 + m = \mu$ the greatest bending-moment in the strut,

or
$$\mu = \frac{m}{\cos l\sqrt{\frac{F}{EI}}},$$

$$f_c = \frac{m}{Z\cos l\sqrt{\frac{F}{EI}}} + \frac{F}{A},$$

$$f_t = \frac{m}{Z\cos l\sqrt{\frac{F}{EI}}} - \frac{F}{A}.$$

Taking for numerical example a round bar of steel as before, and taking $d=1$ inch, $l=48$ inches, and writing the angle in degrees to simplify calculation, we have

$$f_c \text{ or } f_t = \frac{10{\cdot}2m}{\cos 2{\cdot}267\sqrt{\mathrm{F}}} \pm 1{\cdot}273\mathrm{F},$$

$$y_1 = \frac{m}{\mathrm{F}}\left\{\frac{1}{\cos 2{\cdot}267\sqrt{\mathrm{F}}} - 1\right\}.$$

If $\mathrm{F}=0$, f_c or $f_t=10{\cdot}2\,m$, $y_1=\frac{ml^2}{2\mathrm{EI}}$ or $7{\cdot}821\times 10^{-4}m$,

Euler's strength or U for this strut is 1577 lb. It is evidently obtainable by putting the quantity inside the bracket equal to infinity.

First take $\mathrm{F}=1500$ lb. Then

$$y_1=\frac{m}{60}, \quad 2{\cdot}267\sqrt{\mathrm{F}}=87^\circ{\cdot}80,$$

$$f_c \text{ or } f_t=266m\pm 1910.$$

m.	f_c.	f_t.	y_1.	f_c or f_t, if $\mathrm{F}=0$.
0	1910	−1910	0	0
1	2176	−1644	·0167	10·2
2	2442	−1378	·0333	20·4
3	2708	−1112	·0500	30·6
5	3240	− 580	·0833	51·0
7·18	3820	0	·1197	73·2
10	4570	750	·1667	102
20	7230	3410	·3333	204
80	23190	19370	1·333	816

So that an endlong load only sufficient by itself to produce a stress of 1910 lb. per sq. inch and a bending-moment which by itself would only produce a stress of 816 lb. per sq. inch, if both act together, produce a stress of 23190 lb. per sq. inch. In fact, the stress due to m alone is intensified more than 26 times.

Similarly with an endlong thrust of 1400 lb. the stress due to the lateral bending-moment m is intensified 11·04 times. With an endlong thrust of 1000 lb. the stress due to m is intensified 3·185 times. With an endlong thrust of 500 lb. the stress due to m is intensified 1·58 times.

It is obvious that in any strut the stress due to m, a lateral bending-moment, is more and more intensified as F approaches Euler's load U. And if m is due to an inaccuracy of loading, h, the effect becomes four times as noticeable in struts of half the length.

From the above example one sees that it was quite possible for a strong man like Samson to exert a sufficient lateral force to produce fracture in the columns of the temple at Gaza.

Professor Fuller's graphical method of dealing with a metal arched rib is not applicable to struts unless we assume $Fy=0$, and this is of course a very wrong assumption. For the same reason, the graphical method cannot be applied to very flat arches, because it assumes that we know the shape of the loaded arch. If Professor Fuller's method were applicable, we could at once deal graphically with a strut of varying section. This case does not usually need to be solved in practice, but if it must be solved the following method will do.

1st. Assume that EI is constant and of its average value, and obtain the shape of the strut as in (10).

2nd. Use this value of y in the term $\frac{F}{EI}y$ of (8); and use the proper value of I expressed as a function of x, and then by mere integration of (8) find the more correct value of y.

By repeating this process we can obtain y more and more accurately.

Problem.

When a strut is loaded laterally (say uniformly), say that it is hinged at the ends; it is possible by applying the endlong thrust F untruly, that is making the resultant thrust pass through a point h inches from the centre of each end, to give such a value to h as will enable the strut to withstand a maximum load F. In this case it is obvious that the greatest compressive stresses are:—

at the ends, $f=F\left(\frac{h}{Z}+\frac{1}{A}\right)$;

at the middle, $$f=\frac{F}{A}+\frac{\frac{1}{4}Wl}{Z}\,\frac{U}{U-F}-\frac{Fh}{Z\cos l\sqrt{\frac{F}{EI}}}.$$

The problem to find for a given strut and given maximum stress f the value of h which will cause F to be a maximum is easily worked out if a numerical example be taken.

It is obvious from this and from other examples that may be taken, that whether a strut is loaded laterally or is imperfectly straight or has other faults which would cause it to break with a load F smaller than U and fA, it is always possible to adjust the loading so that F may be increased. Thus if a strut is being tested and if for small loads we find that it deflects, the load ought to be readjusted in such a way that its application produces the minimum deflexion; and now if the load is increased we shall find that the strut breaks with a much greater load than if we had not adjusted the method of its application. This adjustment of the method of loading ought to be applied also in the building of structures.

It will be seen that I have merely touched on a few of the many very interesting problems on structures to which the general treatment (3) to (30) may be applied. The lateral load required to prevent a straight strut from bending and to cause its breaking-thrust to be fA is evidently small. We can assume, however, an initial want of straightness or an accidental deflexion, and so find how very slight need be the diagonal bracing in the large struts used in bridge-building.

I have to thank Messrs. Field and Clinton, of the Finsbury College, for checking the mathematical work in this paper.

XXXII. *On a Permanent Magnetic Field.*

By W. Hibbert, *A.I.E.E.*, *F.I.C.*[*]

In the electrical laboratory of the Polytechnic Institute, Regent Street, the earth's magnetic field varies so much that it cannot be assumed as a basis for reasonably accurate measurement. For this reason it was formerly my habit to give students a certain bar-magnet as a temporary standard, the number of lines passing out of the bar being determined afresh from time to time. This magnet was an old one, and its varied experience (now described under the term "ageing") had given it an approximate constancy. The constancy was, in fact, good enough to suggest the possibility of getting a really permanent magnet. I therefore made some tentative efforts to ascertain the effect of slight variations in the hardness and temper of steel on its magnetic properties, with the result that I found the subject too great for my resources.

---

[*] Read December 4, 1891.

But the growth of the modern idea of a magnetic circuit suggested the possibility of achieving my purpose in a way that would be largely independent of the peculiarities of different brands of steel, as well as of the various physical conditions caused by differences in tempering.

I therefore provided a short straight bar-magnet with a couple of arched pole-pieces, of such length and sectional area that there was left between them a narrow air-gap of very small magnetic "resistance." A very flat coil of wire thrust in or out of this gap gave electromagnetic impulses whose value was fairly constant. Experience with these simple instruments led me to believe that a more rigorous application of the principle would give a truly permanent magnetic field. How far this has been realized will appear from the considerations which I now have the honour to submit to the Society.

The first design that at all approximated to a closed magnetic circuit consisted of a cylindrical steel rod with hemispherical pole-pieces. For reasons into which I need not go, this has been superseded by the following modification.

Fig.1 is a general view of the instrument. Fig. 2 is a vertical section of the magnetic circuit. In this, N S is a cylindrical steel rod, 2¼ inches long and 1 inch in diameter, attached to two cast-iron pole-pieces. The upper pole-piece, *a a*, is a circular disk, 4 inches in diameter and ⅝ inch thick, carefully bored to fit the upper end of N S. The lower pole-piece, *b b*, is nearly hemispherical, and about ½ inch thick in the wall. This is attached to the steel rod by means of an iron pin, let into N S and screwed into *b b*.

The opposed faces of the pole-pieces are carefully turned in the lathe, so as to leave between them a circular air-gap less than $\frac{1}{16}$ inch wide.

Fig. 1.

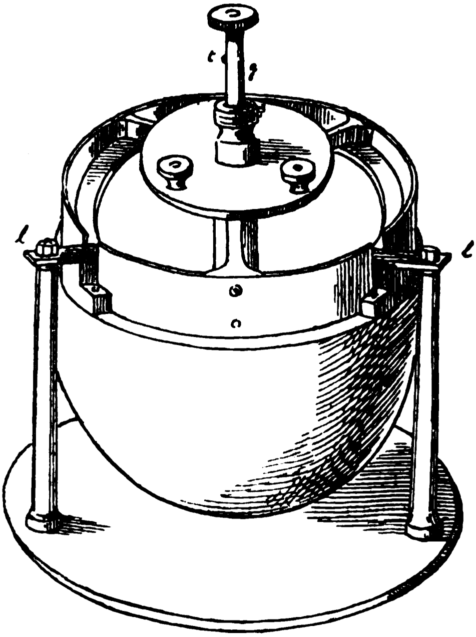
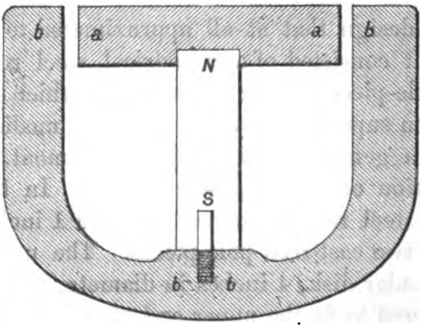


Fig. 2.



The above description gives the form and dimensions of the magnetic part of the instrument. It is magnetized after being put together by a current sent through a coil wound on the steel rod.

The other parts consist of:—

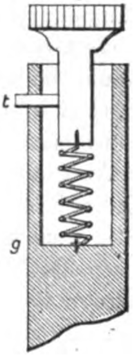
(1) Mechanical arrangements which will rigidly preserve the pole-pieces in position.

(2) Arrangements which will carry and guide a coil of wire as it is thrust through the field in the very narrow air-gap. Screwed to the upper surface of the disk *a a* is a brass casting with three projecting lugs *l l* (fig. 1). These lugs are screwed both to small blocks attached to the hemispherical pole-piece, and also to the tops of three brass pillars (fig. 1).

The coil is wound in a shallow groove cut on the outside of a brass ring, which is turned thin enough in the wall to slide freely through the air-gap. Attached to its upper edge are three arms which radiate from a central boss, the latter sliding up or down on the central guide-rod *g* (fig. 1). The radial arms support an ebonite disk on which are fixed the terminals of the coil.

At the upper end of the guide-rod *g* is an arrangement for allowing the ring to fall whenever the experimenter desires. The upper end of *g* is bored out (fig. 3), and a horizontal slot is cut in the thin wall thus made. In the inner space and connected with the milled head is a rod, from the side of which projects a tooth *t* passing through the slot, whilst from the lower end passes a spiral spring also fixed into the body of *g*. The tooth *t* is thus kept in a definite position, from which it can be moved aside either by a rotation of the milled head, or by the upward movement of the boss connected with the ring. On the inner edge of the boss is a slot through which *t* can pass, but cut slightly on one side of the zero position of the tooth. When an electromagnetic impulse is desired the coil is raised so that the boss, and therefore the ring, rests on *t*. The position of *t* and the height of the ring are so chosen that when the ring is at rest on *t* the coil is altogether above the gap. By a simple rotation of the milled head the coil can be made to fall through the field at any desired moment. The electromagnetic impulse then given to the circuit is, of course, equal to the number of lines inter-

Fig. 3.



linked with the coil during the fall, multiplied into the number of turns of wire in the coil.

Three instruments of the type just described have been made and tested. The means adopted for testing are not the best that could be specified for the purpose. They were chosen because they give reasonable accuracy, and yet allow of a fair number of observations being taken in a limited time.

The method was to compare two throws of a galvanometer-needle, one produced by the discharge of a condenser, the other produced by the magneto-inductor.

The condenser employed was a mica condenser of 0·333 microfarad capacity. The potential difference for charging was obtained from four accumulators, whose electromotive force was determined by comparison with a Latimer-Clark cell, the comparison being made by the potentiometer method.

Having first taken a fair number of observations from the condenser, a corresponding number were taken from the magneto-inductors, the resistance in circuit with each being adjusted till the throw was practically the same as that obtained from the condenser.

I decided on this because it allowed me to use an ordinary damped reflecting-galvanometer. The object being simply to test constancy, it was evident that the subsequent tests would be very much like the first, and that by the above method of working I might neglect damping, and also any consideration of the law of the galvanometer deflexion.

As the observations run through summer and winter they had to be corrected for temperature variation. The coefficients used were those now generally accepted, namely:—For the Latimer-Clark cell 0·077 per cent. per 1° C; for the copper wire of the galvanometer 0·38 per cent. per 1°; and for the German-silver resistances 0·044 per cent. per 1°.

It now remains to give the results obtained. As my present purpose is simply to show how far constancy of field has been obtained, it is not necessary to give many details. Before giving the results it will be proper to state the chief differences between the instruments numbered I., II., and III. in the Table.

Hemisphere No. I.—Air-gap a little more than $\frac{1}{16}$ inch and not quite uniform. Magnetized by flashing current from four accumulators, July 29, 1890. Heated and cooled several times on different days. Very small decay to August 11, on which day the number of lines = 21,007. Next day, August 12, 1890, lines = 21,120. Seven months later, March 14, 1891, lines = 21,035. Accident to coil caused me to take to pieces and magnetize afresh. A current flashed through at a temperature of about 50° C.

Hemisphere No. II.—Most highly finished. Air-gap rather less than $\frac{1}{16}$ inch. Magnetized by flashing. Lines about 34,000. Nearly 50 per cent. greater than in I. This showed tendency to fall. I therefore adopted the method known as "reducing," by sending reverse current through magnetizing coil. Reduction of field about 5 per cent.

Hemisphere No. III.—Magnetic system not so well supported as in two previous instruments. Magnetized by flashing current, and then "reduced" about 20 per cent.

Date.	Temperature of Inductor.	Lines in Inductor Field.		
		No I.	No. II.	No. III.
April 16, 1891......	50° C.	22,030		
" " "	20	21,790		
" 22 "	12·5	21,730		
" 23 "	13·5	21,710	32,360	
May 8 "	16·5	21,710	32,420	
" 23 "	13	21,680	32,330	
" 27 "	13·5	......	......	29,140
" 30 "	16	21,720	32,410	29,290
June 6 "	18	21,720	32,380	29,270
" 12 "	22	21,780	32,470	29,260
" 29 "	21·5	21,720	32,345	29,290
July 10 "	19	21,790	32,510	29,500
" 27 "	20	21,700	32,470	29,550
" 31 "	17·5	21,780	32,460	29,530
Sept. 22 "	16	21,690	32,400	29,470
Nov. 10 "	13	21,700	32,400	29,480
Density of lines in air-gap per sq. cm.	......	515	770	700

The figures in the column headed "No. of Lines" are obtained by the expression

$$N = \frac{100\,CVR}{t} \cdot \frac{d'}{d};$$

where C = capacity of condenser in microfarads,
V = potential difference in volts,
R = resistance in circuit with magneto-coil,
t = number of turns of wire on magneto-coil,
d',d = throws from magneto-inductor and condenser respectively.

The factor 100 translates from practical to absolute units.

In discussing the figures it must be remembered that high accuracy was not attainable at the time of working. The observations were all made in comparatively brief intervals of leisure, and it is likely that the probable error is about 1 in 300 or 400.

There is practically no evidence of magnetic decay in seven months. Such small changes as are indicated point the other way, but I am inclined to attribute most of them to slight inaccuracy of the temperature-correction for the resistances of the circuit.

The two intruments that were reduced by a demagnetizing current (Nos. II. and III.) show a tendency to rise. This tendency is most marked in the instrument which was reduced most. It is evident that only a slight reduction is to be allowed.

No. I., which was magnetized warm, but not "reduced," fell while cooling, and then showed a very slight tendency to increase. It is better than the other two, and is constant enough to be used for any purpose for which an earth inductor is employed.

Nothing is here said of the temperature variation of this magnetic field*. I have made some observations on that point, and found it very small, but as the tests were made with hastily-adjusted appliances, I have thought it better to reserve the matter for further investigation. The temperature range involved in the Table is 10° C., but the figures cannot be used for deducing the temperature-coefficient because of the other sources of error. I am inclined to think that the temperature variation of the condenser (which has not been applied to the calculations) practically neutralizes that of the inductor.

---

* Magnetic field is of course not the right name. The instrument raises once again the question of naming the quantity sometimes called "total field," "total induction," or so many "gausses."

In addition to the evidence of constancy afforded by the Table, there are in my note-books several facts which point the same way. Of these I shall mention only one.

It is well known that magnetic decay is most pronounced just after magnetization, especially if the magnet is subjected to vibration. In several early cases I tested the effect of vigorous blows during and immediately after magnetization, but the evidence of loss was generally very feeble, and in some cases not measurable. In this connexion it may be worth noting that the brass ring which carries the coil is fairly heavy, and that when it falls it produces an appreciable blow. Each of the three instruments has been subject to this shock hundreds of times, but has shown no sign of decay under it.

Perhaps I may be permitted to say that I do not advance the principle of the magnetic circuit as at all novel. It has already been embodied in several well-known applications. But I believe the idea is here made subservient to new purposes.

As a material embodiment of a standard of magnetism (magnetic lines, gausses, or whatever else may be the right name) I find it helpful to many students. For this reason I propose to adjust future instruments of this size to a round number of lines, say 20,000 or 25,000, which will facilitate calculations arising out of their use.

The instrument is a most convenient standard for measuring magnetic quantities, whether it be the lines in any other magnet or the vertical and horizontal components of the earth's field. I am trying to use it for developing a new method of determining these quantities.

It enables me to simplify the magnetometer method for determining magnetic permeability. Over the ordinary magnetizing coil I wind a sufficient number of turns of a secondary.

The coil with its core is fixed in *any* position that is convenient, and the *relative* number of lines determined by the magnetometer deflexion in the usual way. At the end of the magnetization the total *absolute* number of lines (corresponding to the maximum magnetometer deflexion) is found by taking a throw from the secondary coil when the magnetizing current is reversed. This throw is then compared with that obtained from one of the inductors. By this method one is relieved from measuring distances whose cube &c. enters into the formula for reducing the observations.

I ought not to close without acknowledging the help given all through by the instrument-maker, Mr. G. Bowron, one of whose workmen, Mr. Collins, suggested the convenient method of release described in the paper.

PROCEEDINGS
OF
THE PHYSICAL SOCIETY
OF LONDON.

OCTOBER 1892.

XXXIII. *Flexure of long Pillars under their own Weight.*
By MAURICE F. FITZGERALD[*].

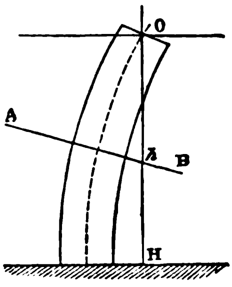
THE origin is taken at the upper end of the neutral axis, abscissæ being reckoned positive vertically downwards, and ordinates horizontal. The flexure is supposed small, and assumed to lie in a vertical plane. The symbols employed are as follows:—

H = total height of pillar;
h = height below top of any point in it;
S = total shear on a section normal to neutral axis;
M = bending-moment;
w = weight of pillar per unit of length;
E and I, as usual, stand for the coefficient of flexural elasticity, and moment of inertia of cross section, respectively.

Taking a plane, A B (fig. 1), normal to the neutral axis, the shear on this plane is the component along it of the weight of the upper part of the pillar (whose top is supposed free); for small bending we have therefore

$$S = wh\frac{dy}{dh} \text{ nearly.}$$

Fig. 1.



---

[*] Read February 26, 1892.

By well-known theorems, $\frac{dM}{dh}=S$ and $M=-EI\frac{d^2y}{dh^2}$, which give by substitution,

$$EI\frac{d^3y}{dh^3}=-wh\frac{dy}{dh}.$$

By writing $\frac{h}{H}=x$ and $m=\frac{wH^3}{EI}$, this takes the form

$$\frac{d^3y}{dx^3}=-mx\frac{dy}{dx},$$

in which, putting $\frac{dy}{dx}=u$, we get

$$\frac{d^2u}{dx^2}=-m\,xu,$$

a differential equation which enters into other questions.

The value of $x\left(=\frac{h}{H}\right)$ runs from 0 at top to 1 at foot of pillar; m has, except for pieces of fine wire a few feet in length, or for very unusually tall and large columns, only a small fractional value in practice.

Integrating the equation $\frac{d^3y}{dx^3}=-mx\frac{dy}{dx}$ in series, we get

$$y=AU+BV,$$

where

$$U=x\left\{1-\frac{mx^3}{2.3.4}+\frac{m^2x^6}{2.3.5.6.7}-\frac{m^3x^9}{2.3.5.6.8.9.10}+\dots\right.$$

V is another series, having x^2 as a factor, and A and B are arbitrary constants.

Calling the first derived function, with respect to x, of U, U′ and so on, the condition of a pillar free at top, and fixed initially vertically to a rigid base is expressed by

$$\frac{dy}{dx}=AU'+BV'=0 \text{ when } x=1, \textit{ i. e.} \text{ at foot,}$$

and

$$\frac{d^2y}{dx^2}=AU''+BV''=0 \text{ when } x=0, \textit{ i. e.} \text{ at top,}$$

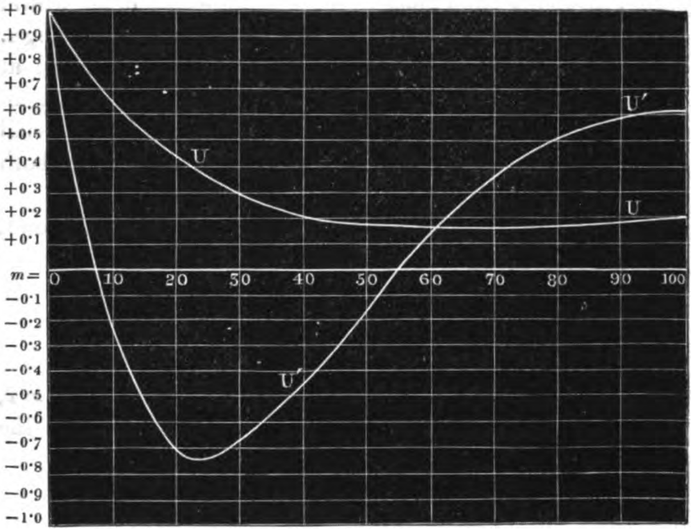
since there is no bending-moment at top.

As V contains x^2 as a factor, the second of these gives $B=0$, and the first then requires $U'=0$ when $x=1$. It will be found, on inspecting the curves plotted in fig. 2, that a value

Fig. 2.

$$y=U_{x=1} \quad \frac{d^3y}{dx^3}=-mx\frac{dy}{dx},$$

$$u=U'_{x=1} \quad \frac{d^2u}{dx^2}=-mxu.$$



of $m=7·85$ nearly is that required. For dimensions in feet, and for steel in which E = about 12,000 tons per square inch, this gives, on putting in the numerical values, and putting L = ratio of length to diameter,

$$H \text{ (in feet)} = \frac{8·1 \times 10^6}{L^2} \text{ for steel tubes,}$$

H being here independent of the thickness, supposed small; and

$$H = \frac{4 \times 10^6}{L^2} \text{ for round steel rods,}$$

as the limiting height of pillar which can stand without bending under its own weight. Thus for $L=100$, the maximum height is about 800 feet, giving a tube 8 feet diameter. For wires, L may be much greater; for instance, the limit at which bending due to its own weight, of wire originally straight and vertical, size No. 28 B.W.G., must occur is about 1·8 feet.

All columns, in practice, naturally fall far within the limits here given. In connexion, however, with the inherent flexibility of very large masses under their own weight, even when direct crushing is prevented (say by external fluid pressure), it may be remarked that for $L = 4$, $H = 47$ miles, approximately; so that a solid steel column 12 miles diameter would bend, even if prevented from bulging, if it were 50 miles high.

The only case of interest, besides that of a column fixed at its base and free at the top, above treated, seems to be that of a heavy upright column, held at top and bottom by external bending-moments so that the neutral axis is vertical at both ends, but otherwise free.

In this case, denoting by suffixes the values at each end, we have

$$AU_0' + BV_0' = 0, \qquad AU_1' + BV_1' = 0,$$

$$AU_0'' + BV_0'' = M_1H^2, \quad AU_1'' + BV_1'' = M_2H^2.$$

V_0' and U_0'' are both zero identically; $U_0' = 1$, and $V_0'' = 2$, which give $A = 0$; $2B = M_1H^2$; and, on substitution in the second and last of the above equations, we get

$$BV_1' = 0 \text{ and } M_1V_1'' = 2M_2,$$

where, in V_1'', the value of m which makes $V_1' = 0$ is to be inserted. The result shows that there is, in this case, a definite ratio between the external bending-moments.

Precisely similar results, as to producing bending, would take place in a bar accelerated by a force applied at its back end, neglecting longitudinal sound-waves; as also to a liquid filament retarded, if it possessed uniform stiffness in virtue of any internal motion.

Belfast, February 1, 1892.

Since the above was in print Prof. A. G. Greenhill, F.R.S., has sent to the writer a paper, published in the Proc. Camb. Phil. Soc. vol. iv. 1881, written by him for Prof. Asa Gray, on the greatest height of poles, masts, and trees, consistent with stability. The differential equation involved is, in Prof.

Greenhill's paper, solved by the aid of Bessel's functions, and the investigation is extended to the cases of a solid cone and a paraboloid of revolution, the general form of the solution for certain other solids of revolution being given. The results for a wire (allowing for a slight difference in the value assumed for E) given by Prof. Greenhill are the same as those above. The function U′ tabulated in the curve fig. 2 appears, from Prof. Greenhill's paper, to be connected with $J_n(\kappa x^m)$ by the relation

$$U' = x^{\frac{1}{2}} J_{-\frac{1}{3}}(\kappa x^{\frac{3}{2}}).$$

Belfast, March 16, 1892.

XXXIV. *Choking Coils.*

By Professor JOHN PERRY, *D.Sc., F.R.S.*[*]

THERE is eddy-current loss of power in all the conducting masses of a choking coil. Hence a choking coil is really a transformer with one primary coil and many secondaries, and much magnetic leakage. In a transformer with many coils, whether or not they have magnetic leakage, it may be shown that any given group of secondaries of given numbers of turns and resistances may be replaced by one secondary without affecting the currents in the other coils; and we may take a choking coil to be a transformer with a primary coil of N turns and resistance R ohms, with C amperes flowing at any instant, the potential difference at its terminals being V, and a secondary coil closed on itself of n turns, resistance r ohms, and current c amperes.

If we assume that the induction per square centimetre β is the same everywhere, and if it follows the law

$$\beta \text{ in C.G.S. units} = \Sigma a_i \sin(ikt + e_i),$$

the average power in watts, wasted in eddy currents in the iron per cubic centimetre is

$$6{\cdot}25 \times 10^{-13} r^2 k^2 \Sigma\, i^2 a_i^2,$$

if the specific resistance of the iron is taken to be 10^4 C.G.S. units. It is less at higher temperatures, being inversely proportional to the specific electric resistance of the iron. The iron is supposed to be of wire of radius r centimetres. Even when we leave the eddy-current loss in the copper out of account, it is to be remembered that the induction is not uniform in the section of a wire, nor is the average induction in each wire the same for all the wires, and therefore the real loss of power in the iron by eddy currents is always greater than the result of applying this formula.

---

[*] Read March 11, 1892.

I am going to assume that one secondary coil with no magnetic leakage may be substituted for all the eddy-current circuits, and this is the same as assuming the truth of the above rule. I ignore magnetic leakage because this is only a preliminary note, and such experiments as have hitherto been made do not enable me to take account of it, for there are no experimental measurements as yet of the want of uniformity of the induction.

The equations of the two circuits are

$$V = RC + N\theta I \quad \text{and} \quad 0 = rc + n\theta I,$$

if I is the total induction (10^8 C.G.S. units being taken as the unit of induction). If

$$A = NC + nc \quad \text{and} \quad q = N^2/R^2 + n^2/r^2$$

(the term n^2/r^2 being really negligible), the fundamental equation for calculating I is

$$A + q\theta I = NV/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Given the law connecting A and I and the resistances and V, I may be calculated, and consequently C and c. Now in ordinary practical transformer calculations A may be neglected in the equation, even with the most complex law of magnetization; and it is this that causes calculations of the induction and secondary currents and voltages in the most complex cases, and even the primary current, unless when there is a small load on the transformer, to be exceedingly easy even when the coils are curiously connected with condensers and choking coils, and when there is much magnetic leakage. But in our present case, it is A itself which is wanted, and another method of working must be adopted. In fact, the value of C does depend very much upon the law of magnetization.

However complicated the magnetic law may be, it can be expressed in the shape :—

If $$I=\Sigma A_i\sigma_i \sin ix,$$

then

$$A=\Sigma A_i\{\sin(ix+f_i)-b_i\sin 3ix+m_i\sin 5ix-\&c.\},$$

x being any quantity which increases continually. To be strictly accurate, even as well as odd harmonics of ix exist in A, and one of my students, Mr. Fowler, has worked them out for some of Prof. Ewing's curves ; but the above formula has been found by Mr. Field to be sufficiently accurate. Of course, instead of ix we may have $(ix+e_i)$ in the above general expression.

When there is no hysteresis, $f_i=0$. When there is constant permeability (no hysteresis and no saturation), not only is $f_i=0$, but $b_i=0$, $m_i=0$ &c., and $\sigma_i=\sigma_1$.

If μ the magnetic permeability of the iron is constant and the magnetic circuit is altogether of iron, as it always ought to be both in transformers and in choking coils, σ_1 stands for $4\pi a\mu 10^{-9}/\lambda$, where a is the area of cross section of the iron in square centimetres, and λ is the average length of the induction solenoids in centimetres,

Equation (1) becomes in the most general case

$$NV/R=\Sigma A_i\{\cos f_i \sin(ikt+e_i)+(\sin f_i+qik\sigma_i)\cos(ikt+e_i)$$
$$-b_i\sin 3(ikt+e_i)+m_i\sin 5(ikt+e_i)\}\ ;$$

and hence, if V or I is given as a periodic function of the time, the other can be found and A and therefore c or C.

If V is a simple sine function of the time, I is so also, with very great, but not perfect accuracy. Assuming that I is a simple sine function, the neglected terms in V can now be calculated. The only problem, however, of importance is the calculation of C assuming that V follows the law $V=V_0\sin kt$.

We may take $q=N^2/R$. Hence

$$-I=(V_0/Nk)\cos kt$$

very nearly, and if $e=n^2\sigma k/r$, being called the eddy-current effect, f being the hysteresis term,

$$C=V_0\left[(1+2e\sin f+e^2)^{\frac{1}{2}}\sin\left\{kt-90^\circ+\tan^{-1}\left(\tan f+\frac{e}{\cos f}\right)\right\}-\right.$$
$$\left.b\cos 3kt-m\cos 5kt\right]\div N^2\sigma k\,.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\qquad(2)$$

We see that the effect of eddy currents without hysteresis is to increase the amplitude of the important term in C, and to produce a lead of $90^\circ-\cot^{-1}e$, whereas the effect of hysteresis without eddy currents is to keep the amplitude unaltered and to produce a lead f. If f is put equal to 0, that is if we assume no hysteresis, we obtain results which seem to be in accordance with such experimental observations as have yet been made.

The effective current $\overline{C}$ (if $\overline{V}$ is the effective voltage), with constant permeability, is $\overline{C}=\overline{V}/N^2\sigma k$. With hysteresis (or with no hysteresis but some saturation of the iron), but no eddy currents, $\overline{C}=1{\cdot}02\,\overline{V}/N^2\sigma k$, taking b as ·2.

With eddy currents and hysteresis,

$$\overline{C}=\overline{V}\sqrt{1{\cdot}04+2e\sin f+e^2}/N^2\sigma k.$$

The average power given to the choking coil or average value of V C is

$$\overline{V}\,\overline{C}\,(e+\sin f)/(1+e^2+2e\sin f),$$

neglecting the small terms due to b and m, and this may be done in all cases where there is not much saturation

Probably there are always traces of the terms in $3kt$ and the higher harmonics in both V and I, but they must certainly exist in either V or I even when there is not much saturation.

It almost seems that in a choking coil we have found what has long been looked for, a method of increasing frequency by mere magnetic means. A condenser shunting a non-inductive part of the circuit would receive currents in which the higher harmonics would be greatly magnified in importance.

To show the magnitude of the terms in (2) I will take a well-known 1500-watt transformer, unloaded, as a choking coil. Here $q=7837$. The total average power wasted in heating the iron being 40 watts, I assume that this is altogether due to eddy currents. Power wasted in eddy currents being $n^2V_0^2/2rN^2$, we have $n^2/r=2{\cdot}117$, when $V_0=283$. An eddy-current coil which would replace all the eddy-current circuits is a coil of 2 turns whose resistance is about 1·9 ohms, short-circuited on itself.

$$e = 0{\cdot}38, \text{ if } k = 600.$$

It is obvious that e is proportional to k and to the square of the radius of the iron wire.

Assuming constant permeability and no eddy currents,

$$C = {\cdot}074 \sin (kt - 90°).$$

With some saturation but no hysteresis,

$$C = {\cdot}079 \sin (kt - 69°{\cdot}2) - {\cdot}0148 \cos 3kt - {\cdot}0037 \cos 5kt,$$

if $b = 0{\cdot}2$, $m = {\cdot}05$.

These values of b and m are usually employed by me for such magnetizations as are common in transformers. When I assume the existence of hysteresis, I take f about 20 degrees.

XXXV. *On the Construction of a Colour Map.*

By WALTER BAILY, *M.A.*[*]

[Plate XI.]

By the term Colour Map I mean a diagram each point of which defines by its position some particular colour. Such a colour map was designed by Clerk Maxwell in the form of a triangle, the angles of which were occupied by certain colours, and all other colours were treated as mixtures of these three primary colours, the composition of the mixture for the colour which occupied any particular point in the triangle being indicated by the length of the perpendiculars from that point on the sides of the triangle.

Now trilinear coordinates, although they afford very elegant methods for the solution of certain problems, are by no means so generally useful or so intelligible as the ordinary rectangular coordinates; and the fact that every colour can be defined by means of a spectrum colour and white light suggested to me the construction of a colour map with rectangular coordinates, in which measurement in one direction should indicate the wave-length of the spectrum colour employed, and measurement at right angles to it should indicate the quantity of white light employed in defining the colour.

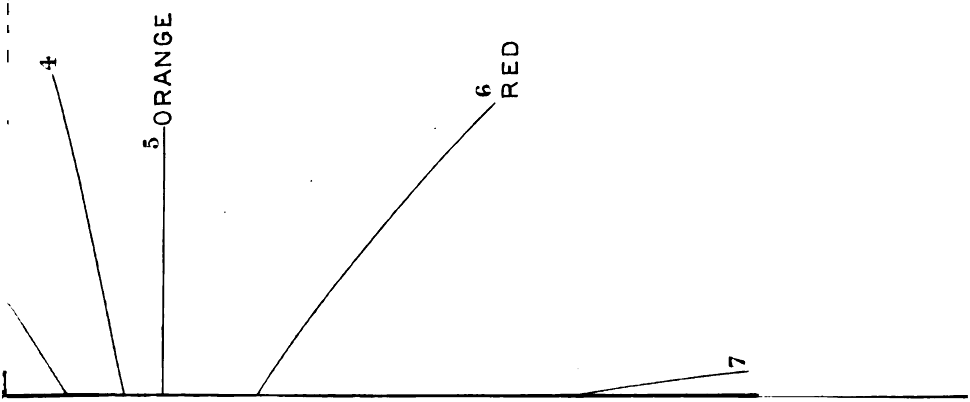
---

[*] Read April 8, 1892.

Let us take a vertical line to represent the spectrum, the lower end giving the red of the spectrum and the upper the violet. The spectrum is supposed to be formed so that equal differences of length measured along the spectrum represent equal differences in the wave-length ; and when the quantity of colour at any point of the spectrum is mentioned, it is intended that a definite small part of the spectrum about that part is to be taken. Now all colours, except the purples, can be formed by adding white light to a spectrum colour. Let the amount of white light required be indicated by a line measured horizontally to *the right* from the proper point in the spectrum. Then the given colour is indicated by the point at the extremity of that line. Again, every colour except the greens has the following property: viz. that if it is added in the proper quantity to some spectrum colour, white is produced. Let the quantity of white produced be indicated by a line drawn from the proper point horizontally to the left. The point at the extremity of this line indicates the given colour. In this way a map is obtained in which every colour has its appropriate position. The greens occur only on the right hand, and the purples only on the left hand, but all other colours, as they can be indicated in both ways, occur on both sides of the spectrum line.

In using the term quantity of white light, I mean that a beam of white light is to be obtained in some definite manner from a definite source of light which forms the spectrum, and that the map is to show how much of this beam is used. Captain Abney finds that the positive pole of the electric arc is a source of light of constant quality, and uses it in his measurements ; and he indicates the quantity of white light used by the ratio between its luminosity and that of the spectrum colour. It is a more complicated matter to express such a ratio than to express the amount of white light only, and I failed to work into a map Captain Abney's method of defining the quantity of white light.

The principle on which this map is founded will come out more clearly by the consideration of fig. 1, which may be considered as a sort of colour staff, to borrow a term from music.


ORANGE
RED

The three horizontal lines represent the three colour sensations—Red, Green, and Violet, with such luminosity that the mixture represented by equal lengths of the three lines represents white light. Thus the vertical lines A, A′, wherever they may be placed, will include between them white light, which will be the more intense the farther they are apart. Any colour whatever may be represented by taking the line A as a base and measuring off the quantities of the sensations to points, R, G, and V. The distances included between A′ and R, G, V give every complementary colour to that represented by R, G, V, and A. The whole of the colours of this system are related together only by the position with respect to one another of R, G, V—that is, only by the differences R G and G V. But if we express the same colours at (say) half the luminosity, we must reduce all these distances to one half, as in fig. 2, and so with any other proportion. It is then not the differences R G and G V, but the ratio of these differences which is constant for all this system of colours. Hence, to determine to what system a colour belongs of which we know r, g, v, the quantities of red, green, and violet sensations respectively, we have only to obtain $\frac{r-g}{g-v}$.

Fig. 1.

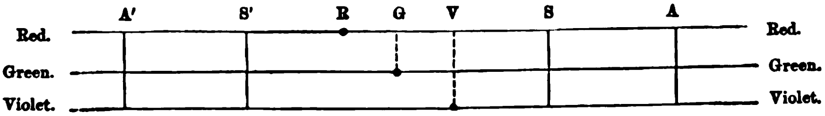


Fig. 2.

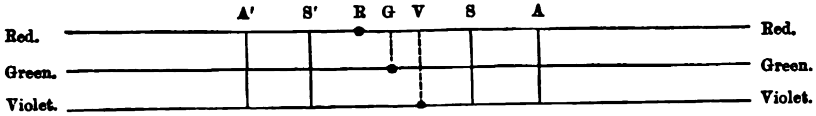
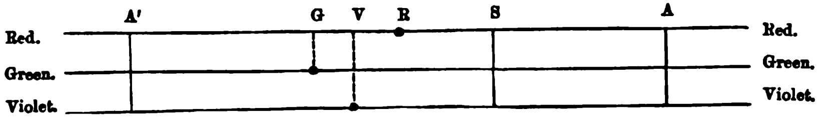


Fig. 3.



In fig. 4, Pl. XI., the vertical line called "spectrum colours" is that along which the spectrum is thrown; and the lines called "line of no Red," "line of no Green," and "line of no Violet" are lines to which distances are to be measured horizontally from any point to show the quantity of red, green, and violet sensations in the colour represented at that point. When these distances are measured from points on the spectrum line, they give the amount of such sensation for the corresponding spectrum colour. The curves which I have used are not intended to represent the true form of such curves, as it is sufficient for explaining the principles of the map that they should be curves having a maximum and shading off on each side. The numbers marked along the "spectrum line" give the value of the fraction $(r-g)/(g-v)$ at each point; and it will be seen that the value is large at the red end of the spectrum, probably beginning with infinity, and diminishes to zero, where the red and green are equal. It then changes sign and remains negative until g and v become equal, when the fraction becomes infinite and again changes sign. For the remainder of the spectrum the fraction continues positive and passes from infinity to zero. The fraction $(r-g)/(g-v)$, which may be called the "Colour Index," has therefore in the spectrum every value from plus to minus infinity, and has all the positive values twice over. Every positive colour index has two spectrum colours:—one in which the order of magnitude of the sensations is Red, Green, Violet, and the other in which the order is Violet, Green, Red. In fig. 1, where the order is that required, let the lines S, S′ give the spectrum colours. Then it is clear that these two spectrum colours are complementary to one another. Also that the colour represented by A is equal to the spectrum colour S plus the colour included between S and A, which is white; and also that the colour at A plus the spectrum colour at S′ form the white between A and S′.

Now suppose the colour index negative, then R, G, V must be arranged in the order R V G or V R G (see fig. 3). We have A, A′, and S, as before; but S′, the second spectrum colour, does not occur, inasmuch as there is no spectrum colour in which green is less than both the red and the violet. Hence the green, which is represented by A, can be defined only by the addition of white to a spectrum colour; and the purple, which is represented by A′, can be defined only by the fact that when added to a spectrum colour they can form white.

To see how what precedes is represented in the Colour Map (Plate XI. fig. 4), take any line perpendicular to the spectrum-line, say the line in the orange for which the Colour Index is 1·0, and compare this line with fig. 1. S is the point on the spectrum-line, V is the point at which the "line of no violet" is crossed, and G and R the points in which the lines of no green and no red respectively are crossed, and S′ represents the complementary spectrum colour, which is represented on the thick line at the point marked 1·0. This thick line, along which the figures are marked, represents the spectrum which is complementary to that from Red to Yellow, and itself extends from Violet, of which the colour index is zero, to Blue, of which the colour index is infinite. A similar line gives the complementary spectrum of the part from Blue to Violet, and itself extends from Red when the colour index is infinite to Yellow when it is zero. The region on the right outside *all* the lines gives all the colours to be obtained by adding white to a spectrum colour; and to ascertain the amount of each sensation, we have only to measure horizontally to the line giving the zero of that sensation. The region on the left outside *all* the lines gives all the colours capable of making white with spectrum colours; and here, again, to ascertain the amount of each sensation we have only to measure horizontally to the line giving the zero of that sensation. It will thus be seen that the whole map is really constructed on one single principle. It is obvious that if a series of colours are obtained by some definite law, their positions on the map will lie on some line straight or curved.

It remains to consider the spaces enclosed within the lines. On the right between the spectrum-line and the nearest sensation zero-lines lies a space which has a real meaning, as the points in it represent colours in which the sensations have certain positive ratios to one another; but these ratios give a more intense colouring than the spectrum colours themselves, and therefore such points cannot represent any colours which can be seen by a normal eye, because, as was known to Newton, every mixture of colours is more diluted than the spectrum colour which it most nearly resembles. This region may be called an abnormal region. The colours it represents would be visible to eyes more or less colour-blind. There are two abnormal regions on the left of the figure between the complementary spectrum-lines and the red and violet zero-lines respectively.

The remaining portion of the map, viz. that lying *between* zero sensation lines, is of a different nature. At any point in this region the distances measured to the zero lines are not all in the same direction; so that one or two out of the three sensations must be considered to be negative. As no one possesses a negative colour sensation, the colours represented in this region are imaginary. This may be called the imaginary region. Though it has no physical meaning it will be found to have its value in connexion with the geometrical structure of the map. As an example of this, consider the complementary spectrum-lines. They end abruptly, leaving a gap opposite the green; but they may be continued across the gap in such a way as their general form seems to point, and this has been done in fig. 4, by continuing the complementary spectrum-lines until they meet in a cusp at the point on the right marked $-1{\cdot}0$. This extension lies wholly in the imaginary and abnormal region, and may represent the missing complementary spectrum of green.

The map affords convenient methods for calculating the effect of mixing colours. Let a colour which has the sensation red, green, and violet in the proportion r_1, g_1, v_1 be represented by $r_1 \mid g_1 \mid v_1$. Then, if we take two colours $r_1 \mid g_1 \mid v_1$ and $r_2 \mid g_2 \mid v_2 \mid$, the mixture of these colours in the proportions l_1 and l_2 will give the result $l_1r_1+l_2r_2 \mid l_1g_1+l_2g_2 \mid l_1v_1+l_2v_2$. The index of this colour is

$$\frac{l_1(r_1-g_1)+l_2(r_2-g_2)}{l_1(g_1-v_1)+l_2(g_2-v_2)}.$$

Let the spectrum colour having the same index be $r \mid g \mid v$. In order to find the quantity of white which must be added to this spectrum colour to produce the required colour, it is necessary that the luminosity of the colour should be altered to that luminosity at which the colour is represented in the map. This can be done by multiplying the coefficient of each sensation by the fraction $(r-g)/\{l_1(r_1-g_1)+l_2(r_2-g_2)\}$ or one of the equivalent fractions. The resulting sensation coefficients are

Red . . . $(l_1r_1+l_2r_2)(g-v)/\{l_1(g_1-v_1)+l_2(g_2-v_2)\}$.
Green . . $(l_1g_1+l_2g_2)(v-r)/\{l_1(v_1-r_1)+l_2(v_2-r_2)\}$.
Violet . . $(l_1v_1+l_2v_2)(r-g)/\{l_1(r_1-g_1)+l_2(r_2-g_2)\}$.

The coefficient of the white to be added to the spectrum colour is obtained while the colour lies on the right hand by subtracting r from the red coefficient above obtained, or by subtracting g and v from the other coefficients respectively. When the colour lies on the left hand, the white is obtained by adding r to the above red coefficient, or g and v to the green and violet coefficients respectively.

I have applied the formulæ given above to obtain the curves showing the results of adding together in any proportion two spectrum colours so related to each other that if the first is $r \mid g \mid v$ the second is $v \mid g \mid r$. The index of the second colour being $\frac{v-g}{g-r}$ is the reciprocal of the index of the first.

The curves obtained are shown in fig. 5. Consider the curve numbered 2. This is the locus of mixtures of the blue whose index is 2·0, and the yellow whose index is 0·5. The curve passes through these two points of the spectrum, giving the cases in which a zero quantity of one of the colours is taken; and every other mixture is indicated by some point on the curve joining these two points and lying to the right of the spectrum-line. In this figure the dotted horizontal lines occupy the positions where the indices are zero and infinity respectively, so that the portion of any curve which lies outside of them must be repeated again on the left side of the complementary spectrum-line. In curve No. 2 two small parts do lie outside the dotted lines, and, accordingly, these two parts are repeated to the left of the complementary spectrum-line. We have then the curve No. 2 in three separate portions, which it is not possible to connect physically, as the missing part of the curve lies in the imaginary and abnormal regions. But what is not possible for physics is easy for geometry. We cannot subtract one spectrum colour from another, but we can subtract the lines representing the sensations in one spectrum colour from the lines representing the sensations in the other spectrum colour; and so by subtracting one spectrum colour from the other in any proportions we can complete the curve No. 2 through the imaginary and abnormal regions and so obtain the complete and continuous curve. Curves Nos. 1 and 0 have no portion on the complementary side, but curves Nos. 3 and 4 have a considerable portion on that side. A new feature is shown when we take the locus numbered 5. This is got by combining the spectrum indigo, having index 1·0, with spectrum orange, having the same index. These are complementary colours. When added together in the proper proportion they produce white, and when added in any other proportion they produce white plus whichever spectrum colour predominates. Hence the locus consists of horizontal straight lines through the two points in the spectrum-line, going off to infinity, where the colour indicated is white, considered as a spectrum colour infinitely diluted with white light. Next consider curve No. 7. The main portion of the curve lies to the left, and starts from points in the complementary spectrum-line which indicate the spectrum colour chosen. The parts of this portion which lie outside the horizontal dotted lines are repeated to the right of the spectrum-line; the remainder, obtained by subtraction, lies wholly in the imaginary and abnormal regions. All these curves pass through a certain pair of points, as may be easily shown.

The first spectrum colour is . . . $r \mid g \mid v$.
The second is . . . . $v \mid g \mid r$.

By subtraction of one from the other, we get a colour

$$r-v \mid \text{zero} \mid v-r.$$

The resulting colour has therefore no green, and has the red and violet equal in amount but opposite in sign. These conditions are satisfied at the two points shown in the figure.

In this figure the lines are drawn under the condition that the index of one spectrum colour is the reciprocal of the index of the other; but any number of other systems of lines might be drawn showing combinations of two spectrum colours, so that it is evident that every colour can be resolved into two spectrum colours in an infinite number of ways.

There are three regions in fig. 5 which are shaded to show that none of the curves pass through them. These regions might probably be filled up by curves drawn through points in the imaginary part of the complementary spectrum to which I have already alluded.

Now the complementary spectrum-line and the curves giving mixtures of two spectrum colours have been drawn by strict arithmetical methods from certain curves of hypothetical form which indicate the intensity of the sensations for each point of the spectrum ; but they can also be plotted out by direct experiment.

To plot out the complementary spectrum-line, add to a spectrum colour its complementary until white is produced, measure the quantity of white, and mark off a horizontal line to the *left* from the point in the spectrum of a length proportional to the quantity of white. The end of this line is a point in the complementary spectrum ; other points may be obtained in the same way, and the normal part of the complementary spectrum-line be drawn.

To plot out the curve giving the mixtures of two spectrum colours, take a third spectrum colour and make a colour patch of the first two colours, and another colour patch of the third colour and white. Keep the luminosity of the third colour constant, and vary that of the other colours and the white until both patches are of the same colour. Then measure the quantity of white used and mark off a line from the position of the third spectrum colour to the right proportional to the quantity of white. The end of this line gives a point in the curve. By taking other spectrum colours as the third colour other points may be obtained. If, however, it is found impossible to make the two patches of the same colour, then throw the three spectrum colours together, and keeping the luminosity of the third colour constant vary that of the other two until the three produce white ; measure the quantity of white, and mark off to the left from the position of the third spectrum colour a line proportional to the quantity of white obtained. The end of this line is a point in the curve. If both these methods fail the point on the curve corresponding to the third spectrum colour lies in the abnormal or imaginary regions, and cannot be determined by experiment.

When the derived curves have been plotted out by experiment, it will be possible to modify the hypothetical forms of the curves of intensity of the sensation in the spectrum so as to make the curves derived from them accord more closely with the results of experiment, and so to arrive by gradual approximation to the true form of those curves.

XXXVI. *A Note on the Electromotive Forces of Gold and of Platinum Cells.* *By* E. F. HERROUN, *Professor of Natural Philosophy in Queen's College, London*[*].

IN nearly all modern text-books of Physics the metal platinum is placed after gold in Volta's Electropositive Series. This no doubt is partly owing to the well-known fact that gold is attacked by chlorine or nitrohydrochloric acid more readily than platinum, and it might therefore be reasonably supposed that gold evolves more heat in the formation of its chloride than does platinum. On referring to the values for the heats of formation of the chlorides of these two metals, as given by Julius Thomsen[†], one finds, however, that the heat attending the formation of auric chloride is, per equivalent, only about half as great as that in the case of platinic chloride.

Assuming that the voltaic constants of metals are deducible from the thermochemical values of their compounds, the above facts would compel us to regard gold as more negative than platinum, at least when immersed in chloride solutions. (The same observations would also apply if oxygen were the attacking medium, as Thomsen gives the heat of formation of platinic hydrate as a considerable positive number, while that of auric hydrate is a large *negative* quantity.)

It was, therefore, an interesting point to determine how far the actual electromotive forces obtained with gold and with platinum agreed with these conclusions, and I endeavoured to find records of the electromotive forces of cells in which these metals are immersed in solutions of their chlorides opposed to some other metal in a solution of its corresponding salt. While there are many references to the E.M.F.'s set up in single fluid cells in which platinum is one of the metals used, the references to gold are scanty, and even with platinum I have only succeeded in finding one recorded measurement in which the platinum was immersed in a solution of its own salt. This was a measurement made by Wheatstone*, in which liquid zinc amalgam was opposed to platinum in a solution of platinic chloride. He found in measuring the E.M.F. of this cell that it required 40 turns of his rheostat, as compared with 30 turns required for a form of Daniell cell. Now, assuming his Daniell cell to have had an E.M.F. of 1·09 volt, the value in volts for the zinc amalgam, platinum-platinic chloride cell would be 1·453.

---

[*] Read March 25, 1892.

[†] *Thermochemische Untersuchungen*, iii. pp. 412 & 430.

In a list of the potential differences between different metals and graphite simply immersed in water, Götz and Kurz† give the values 0·48 volt for gold and 0·37 volt for platinum, the value for zinc and graphite being 1·37 volt. This would make platinum more electronegative than gold by 0·11 volt; but these values cannot be accepted as in any degree expressing the actual electromotive forces concerned.

Exner and Tuma‡, on the other hand, taking carbon $=0$, give $Pt = 0·05$, $Au = -0·05$, which appear to be much more probable values, and make gold, as its thermochemical data require, negative as regards platinum.

Under these circumstances it appeared worth while measuring the actual E.M.F. set up between zinc and gold, and zinc and platinum, in solutions of their own chlorides of equal molecular strength.

Zinc-Platinum Cell.

The heat of formation of $PtCl_4$ is apparently unknown, but as it appears to be impossible to prepare a *neutral* solution of that salt, compounds such as $PtCl_4,2HCl$ or $PtCl_4,2NaCl$ must be substituted, and their heats of formation are given by Thomsen. I selected the latter salt on account of its freedom from acid, and prepared a neutral solution having the strength of $\cdot 25(PtCl_4 2NaCl)100H_2O$, which therefore contained about 2·75 grams of Pt in 100 cub. centim. of solution.

---

* Wheatstone's Scientific Papers, p. 115.

† *Electrotechnie Zeit.* ii. p. 30.

‡ *Wien. Ber.* xcvii. p. 917.

Thomsen gives for [$Pt, Cl_4, 2NaCl$, aq] the value 73720 + 8540 = 82,260 calories; and for [Zn, Cl_2, aq] the value 112,840 calories. These numbers would give as the heat of replacement of one equivalent of platinum by zinc the nett heat evolution of 35,855, which is equivalent to a theoretical E.M.F. of 1·548 volt.

A cell was set up consisting of an amalgamated pure zinc rod immersed in a solution of $\cdot 25\ ZnCl_2 100H_2O$, opposed to a clean platinum plate immersed in the solution of sodio-platinic chloride above described. The two solutions were separated both in this and other experiments by an ordinary porous earthenware pot, and the E.M.F. was measured by balancing it against a difference of potential by Poggendorff's method.

The standards taken were a Latimer-Clark cell, which was assumed to have an E.M.F. of 1·435 volt at 15° C., and a chloride of silver battery (modified De-la-Rue cell), which by comparison with the Clark cell was found to have an E.M.F. of 1·045 volt. I find this cell more convenient in using Poggendorff's, or any similar method, as its E.M.F is not appreciably disturbed by its sending a small current, or by shaking, and it has a smaller temperature-coefficient than the Clark cell. The temperature of all the cells used in these experiments only varied between the narrow limits of from 12° to 15° C.

In one experiment the zinc-platinum cell, when first set up, gave an E.M.F. of 1·647 volt. It was then allowed to send a current through a low external resistance for five minutes, and after further resting for five minutes its E.M.F. was again measured, when it was found to have dropped to 1·473 volt. After a further rest of about ten minutes it recovered to 1·507 volt, at which value it remained tolerably constant. During the passage of the current from the cell the platinum plate became covered with a black deposit of finely divided platinum, and I thought it not improbable that this alteration of the surface might be the cause of the marked falling off in E.M.F.; but on removing the solution surrounding the platinum plate and replacing it with fresh, the E.M.F. regained its initial high value although the surface of the plate had not been disturbed. It appeared from this that the high initial value was probably due to oxygen dissolved in the liquid, which the platinum would be very apt to occlude superficially, and which would account for the uncertain values of the E.M.F.

Whatever may be the cause, this variability renders the measurement of the E.M.F. of this form of cell very difficult.

In other experiments values as high as 1·7 volt were obtained on first setting up the cell, which, after sending a current and resting, fell to the tolerably stable value of about 1·525.

Maximum	E.M.F.	= 1·70	volt.
Minimum	,,	= 1·473	,,
Average	,,	1·525	,,

The average value (1·525) is seen to be slightly lower than the E.M.F. calculated from the thermochemical equation (1·548); but the difference is small (·023 volt), and is well within the limits of experimental error with such a variable cell. There is therefore no reason to assume that its actual E.M.F. departs from the theoretical value.

Zinc-Gold Cell.

The heat of formation of [Au,Cl_3, aq] is given by Thomsen as 27,270 calories, and that of [Zn,Cl_2, aq] being 112,840, the difference per equivalent gives 2·044 volts as the theoretical E.M.F. of zinc, displacing gold from weak solutions of its chloride.

A cell consisting of an amalgamated zinc rod immersed in a solution of zinc chloride having the strength of $\cdot 25ZnCl_2 100H_2O$ opposed to a plate of gold in a solution of auric chloride of equal molecular strength, was set up and its E.M.F. immediately measured. It was found to give an E.M.F. of 1·855 volt, and after actual short-circuiting for five minutes it had only fallen to 1·834. These values were in fact the extreme limits of the variations that I have observed on repeating the experiment, and the constancy of this cell contrasts in a striking manner with the variability of the platinum cells.

Maximum E.M.F. = 1·855 volt.
Minimum „ = 1·834 „
Mean „ = 1·844 „

Assuming the thermal values for gold to be accurate, the actual E.M.F. thus measured is seen to be ·2 volt below the calculated value; or, adopting the convention suggested by Messrs. Wright and Thompson, —0·2 volt is the *thermo-voltaic constant* for gold in dilute neutral solutions of its chloride.

When a platinum plate was substituted for the gold plate and immersed in the gold-chloride solution, the E.M.F. of the cell thus formed was found to be 1·782 volt, *i. e.* less than the gold-gold chloride, but greater than the platinum-platinic chloride cell given above.

From the thermochemical values one might conclude that platinum would be capable of replacing gold from gold chloride; but, so far as my experiments have gone, I have not found this to be the case, nor on the other hand is gold able to replace platinum from platinic chloride, which, of course, is not to be expected.

When a plate of gold and another of platinum are immersed together in pure water or in dilute hydrochloric acid, the gold acquires slightly the higher potential. If strong hydrochloric acid be substituted for the dilute, the direction of the difference of potential becomes doubtful, and on the addition of nitric acid to the strong hydrochloric, so as to form *aqua regia*, the platinum acquires distinctly the higher potential, and if the outside circuit be closed on a galvanometer, a very decided current flows from the gold to the platinum through the cell (*cf.* Ganot's Physics, article on "Electromotive Series").

Why the nascent chlorine combines readily with the gold where it evolves but little heat and slowly and reluctantly with the platinum, in which reaction much more heat is disengaged, is a problem which at present appears to admit of no satisfactory solution.

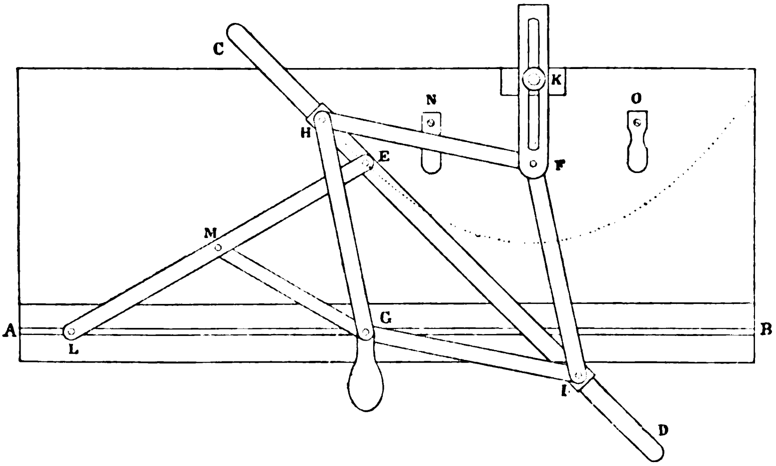
XXXVII. *On an Instrument for Drawing Parabolic Curves.* *By* RICHARD INWARDS, *F.R.A.S.**

THIS instrument, which I have now the pleasure of showing to the members of the Physical Society, is designed for the purpose of drawing by one simple operation any parabola of short focus, such as might be wanted for setting out the curve of a lamp-reflector, or in making a diagram of the path of a comet or of a projectile.

It is based on that property of the parabola by which any point in it is equally distant from its focus and from the nearest part of its directrix.

In the diagram, which represents the instrument as mounted on a drawing-board, F is the focal pivot, which is adjustable for larger or smaller parabolas by moving the piece supporting it in a slide to which it can be clamped by the screw K.

AB is a slot formed by two steel straight edges with a narrow opening between them. The centre line of this slit is the directrix of the parabola to be drawn.



Along the slit AB travel two pins L and G, which are parts of a frame or system LMEG of such a nature that LM, ME, and MG are equal to each other, so that in any position E is compelled to be vertically over G.

A parallel frame HGFI is so constructed that one corner rotates on F and the opposite corner rotates on G. The diagonal bar CD can travel along between guides at H and I.

---

* Read May 13, 1892.

A pencil or scriber is fixed at E, and the paper is placed under this, and held by the clips N and O.

A handle is provided at G, on moving which along the line AB it will be seen that the pencil E is compelled to describe a curve such that it is always equally distant from F and from G, and as G is on a straight line and F is a fixed point, that curve must be a parabola.

The fact that any point in the diagonal of a rhombus must be equally distant from one pair of its opposite angles is obvious on a moment's consideration.

The centre line of the diagonal bar CD is always a tangent to the parabola which is being constructed. The curves so drawn may be used for the production of templates for lenses or mirrors, and they could be drawn small and then magnified either by photography or by a pantagraph arrangement.

This instrument is a combination of well known link movements, but I do not think they have ever before been applied to the production of a parabolic curve from a single straight line motion.

The instrument may be so constructed that any play between the sliding pins and the slot may be avoided by pressing the handle down towards the lower side of the slot, which thus becomes a ruler. In the event of any machine being constructed on this principle, gravity itself might make this pressure.

20 Bartholomew Villas, Kentish Town, N.W.,
February 2, 1892.

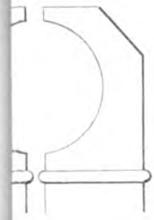
XXXVIII. *A Portable Instrument for Measuring Magnetic Fields. With some Observations on the Strength of the Stray Fields of Dynamos.* *By* EDWIN EDSER *and* HERBERT STANSFIELD*.

[Plates XII. & XIII.]

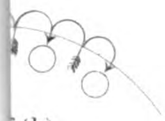
THIS instrument was constructed for the purpose of giving direct readings for the strength of magnetic fields, such as are found in the neighbourhood of Dynamos; thus avoiding

---

* Read May 13, 1892.



ig. 7.



5(b).

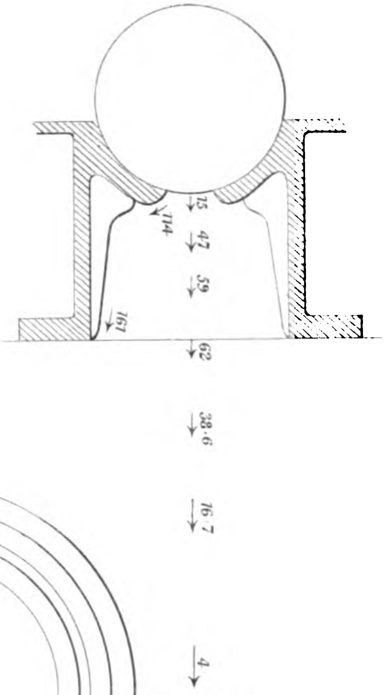


Fig. 5(c).

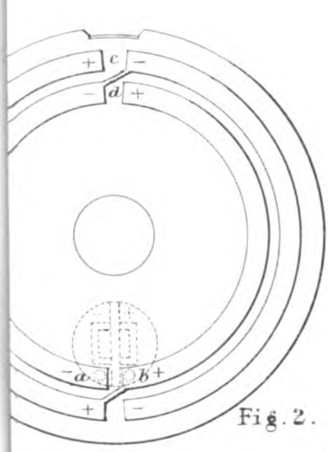
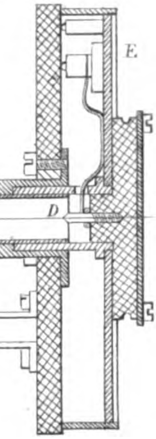


Fig. 2.



the inconveniences attending the Ballistic method. Portability, a considerable range, and a fair degree of accuracy were the qualities sought. The instrument, as now constructed, whilst satisfying the first of these conditions, may be used to measure any field from 1 line per centimetre upwards, with an error of only about 2 per cent.; the only accessories required being a dry cell and a resistance-box.

In principle it is the inversion of the D'Arsonval galvanometer; the torsion necessary to restore a coil, through which a constant current circulates, to its normal position, parallel to the direction of the lines of force, furnishing readings proportional to the field at the position of the coil*.

A diagram of the instrument is shown in fig. 1 (Plate XII.). A B is a small coil, oblong in shape, wound of No. 44 B.W.G. copper wire, and suppported half on each side of a sheet of mica. It is suspended from each end by strips, 10 centim. long, of rolled German-silver wire, each strip having a loop which is passed over a small brass hook riveted on to the mica, and in electrical communication with a terminal of the coil. The strip C A is in electrical connexion with the case of the instrument at C, whilst the strip D B is insulated from it at D by an ebonite plug, attached to the torsion-head E. Inside this latter is a commutator for automatically reversing the current, so as to take readings on each side of the zero. It consists of four semicircular strips of copper, cross connected as shown in fig. 2, *a* and *b* being connected to the two battery terminals. Two springs, one soldered to the case, the other insulated from it, but connected to the end of the suspending strip D B, press on these semicircular strips. When the torsion-head is at zero no current passes, the springs then being at *c* and *d* respectively (fig. 2). To take a reading the torsion-head is turned, thus sending the current through the coil. Should the latter be deflected in the wrong direction, the current can be reversed by means of the plug contact, P, attached to the battery leads. Readings are taken on each side of zero in order to eliminate any error due to imperfect balancing of the coil; an aluminium pointer G, attached to the coil, being always brought by the torsion to the zero position on a small scale.

---

* After the completion of this instrument our attention was called to some experiments of Messrs. Siemens and Halske, in which the same principle was used.

"In order to measure the intensities of the rotary field, a coil was hung in the centre of the ring, in such a way that its magnetic axis was perpendicular to the measured direction of the resulting magnetic axis of the ring. The coil was then excited by a continuous current, and was kept in position by a spring. The torque of the spring served as a measure of the intensity." "Deduction and Experiments on Rotary Currents." A. du Bois Reymond, 'Electrical Review,' June 5, 1891, vol. xxviii. No. 706.

In order to obtain at once a spring-suspension and an adjustment for the torsion of the strip, a particular form of geometrical slide is used. A A′ (fig. 3) is a thick brass tube, turned at B B′ to a slightly conical plug to fit the tube of the instrument (fig. 1). Two grooves (seen in plan at C C′) are made along this tube, a cross head F (fig. 1) on the screw E F fitting into them. This screw is drilled along its whole length to admit a thick wire H I C, the latter having a cross head I, also fitting into the longitudinal grooves. These two cross heads are then connected to the two ends of a spiral spring, in such a manner that they are pressed by it against opposite sides of the grooves. The suspending strip being connected to the central wire H I C at C, its tension can be increased or diminished by means of the nut K, without altering the position of the coil. Any sudden jerk will also be taken by the spring, thus obviating the risk of stretching the suspending strip. Scratches on the wire H I C near H indicate the tension used.

As a source of current a Hellesen dry cell is used. When joined up through 50 ohms the E.M.F. of the cell is practically constant, whilst its internal resistance is negligible*.

The resistance of the instrument having been made up to 50 ohms, it follows that its sensitiveness can be varied by introducing an independent resistance in the circuit.

Let C = constant of instrument (*i. e.* field for 1° of torsion, with no external resistance in circuit);

---

* See *Electrotechnische Zeitschrift*, August 1, 1890, vol. ii. No. 31. Republished in pamphlet form by Siemens Bros. and Co., Ltd. We have independently verified these results.

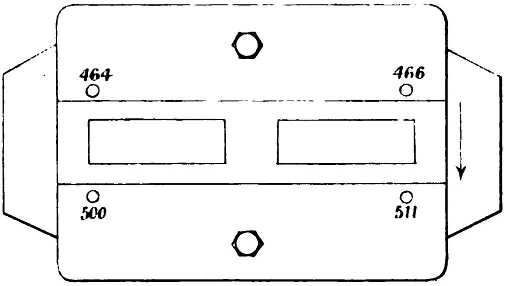


Fig. 3.

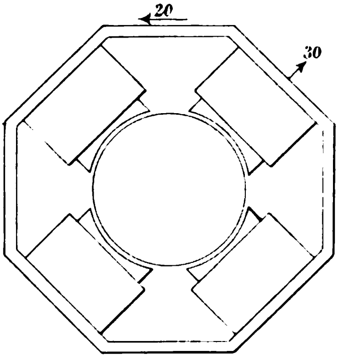
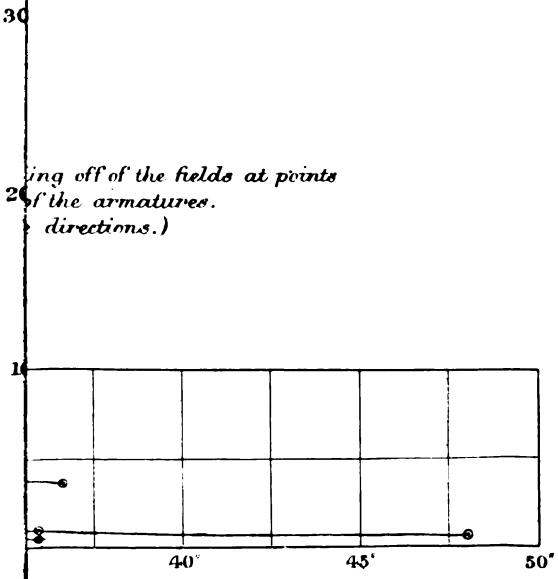


Fig. 4.



ing off of the fields at points
f the armatures.
directions.)

n = multiple of 50 ohms in circuit, exclusive of resistance of instrument;

θ = mean angular torsion;

then

Field in C.G.S. measure $= C\,(n+1)\,\theta$.

C was determined, and the instrument calibrated, between the coils of a galvanometer of the Gaugain type through which a known current was passed. For an E.M.F. of 1·45 in the dry cell it was found to be ·293. The error shown in the calibration was always below 2 per cent.

By permission of the Committee of Experts, and of a number of the firms exhibiting, a series of measurements were made at the Electrical Exhibition, Crystal Palace. The results obtained are shown in the remaining figures.

Fig. 1 (Plate XIII.) shows the fields measured at various distances from different dynamos, the distances plotted as abscissæ and the fields as ordinates. It is noticeable that machines of the multipolar type show a much steeper curve than other dynamos. This is especially noticeable in the case of the Gulcher Dynamo curve (G).

Fig. 4 (Plate XII.) shows the fields round Mr. Kapp's 8-pole machine. They are noticeably small.

Fig. 2 (Plate XIII.) shows the effect of edges, corners, &c. on the strength of field. On the flat surface of the pole-piece the field was about 600 (varying between 517 and 690), on the edges increased to about 1000, whilst on the corners it reached a strength of over 1100 C.G.S. lines per centimetre.

Fig. 3 (Plate XIII.) shows the deformation of the stray field produced by the armature reactions. The measurements were made on an Elwell-Parker Motor. The strength of field on the trailing edge was about 460, whilst that on the leading edge was about 500.

Fig. 6 (Plate XII.) shows various measurements made on the Gulcher Dynamo; Fig. 7 (same Plate) gives the field near one of Mr. Kapp's Dynamos; Fig. 4 (Plate XIII.) shows the fields at two positions of a Laing, Wharton, and Down shielded dynamo.

Some curious effects of armature reactions are noticeable on the Thomson-Houston Dynamo (fig. 5, Plate XII.). As the bars in this machine act as a yoke, the result is due to combined magnetic leakage and armature reaction.

By the kindness of Mr. Harrison we were enabled to make some experiments on a watch, previously unmagnetized, which he lent us. We found that a field of about 10 had no appreciable effect on its rate of going, but that after being subjected to a field of about 40 it lost about 8 minutes per day; and even after being demagnetized in an alternating field it still continued very erratic in its actions. Of the dynamos whose fields we have measured, with the exception of the Thomson-Houston, Ship's Dynamo (Laing, Wharton, and Down), and Mr. Kapp's large Multipolar, it would not be safe to go nearer than about 2 feet[*]. Moreover, with a watch with a steel balance-wheel (the one experimented upon had a brass one), even greater precautions might have to be observed.

Finally, we wish to record our thanks to Mr. Harrison for allowing us to experiment on his watch; to Mr. Barton for his assistance in constructing the instrument; and to Messrs. Crompton, Kapp, Laing, Wharton, and Down, the Gulcher Company, the Electric Construction Corporation, and Major-General Festing, for permission to experiment on their various dynamos, and also to publish the results.

XXXIX. *Note on the Measurement of the Internal Resistance of Cells.* *By* E. WYTHE SMITH[†].

In order to determine the actions which take place in an accumulator during charge and discharge, it is necessary to know the working electromotive force at the different stages. This might be observed by breaking the circuit; but immediately on doing this the electromotive force varies at a very rapid rate, so that if only four or five seconds be occupied in taking the measurement an error of 25 per cent. may be made in the difference between the electromotive force and the terminal potential difference. If time-readings be taken after breaking the circuit and a curve drawn connecting E.M.F. and time, this curve may be produced back in the way described by Prof. Ayrton and others in a paper at the Institute of Electrical Engineers. But, as this method has its objections in addition to that of interrupting the circuit, it is very desirable to determine the actual working E.M.F. in some other way.

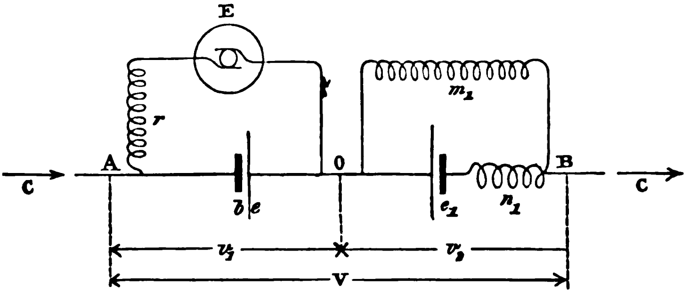
---

[*] One could not safely go within three feet of the Elwell-Parker Continuous Current Transformer.

[†] Read June 24, 1892.

The E.M.F. could be readily obtained from the terminal P.D. and the current if the internal resistance were known. It is for the determination of this latter quantity that I have devised the following modification of Mance's method. Of course this method is applicable to the measurement of the resistance of other forms of battery besides accumulators.

Fig. 1.



In fig. 1 let the cell of E.M.F. e, and internal resistance, b, be the one experimented upon, r being the resistance of the external circuit which may contain an E.M.F. E, for example that of the dynamo used to charge the cell. This circuit is connected at O with an auxiliary circuit, in which the resistances m_1 and n_1 are so adjusted that the points A and B are at the same potential, the resistance of the cell of E.M.F., e_1, being included in n_1. Suppose a current, C, from some external source to pass through both circuits in series. The P.D. between A and B will now be V. Let P.D. between A and O be v_1, and that between O and B be v_2, $(v_1+v_2)=\text{V}$. We have

$$\text{C}=\frac{v_1+e}{b}+\frac{v_1+\text{E}}{r}=\frac{v_2-e_1}{n_1}+\frac{v_2}{m_1};$$

therefore

$$v_1=\frac{br\text{C}-re-b\text{E}}{b+r},$$

and

$$v_2=\frac{m_1n_1\mathrm{C}+m_1e_1}{m_1+n_1},$$

$$\mathrm{V}=\frac{b+\mathrm{C}-re-b\mathrm{E}}{b+r}+\frac{m_1n_1\mathrm{C}+m_1e_1}{m_1+n_1}.$$

But when $\mathrm{C}=0$, $\mathrm{V}=0$,

$$\therefore\ 0=\frac{-re-b\mathrm{E}}{b+r}+\frac{m_1e_1}{m_1+n_1};$$

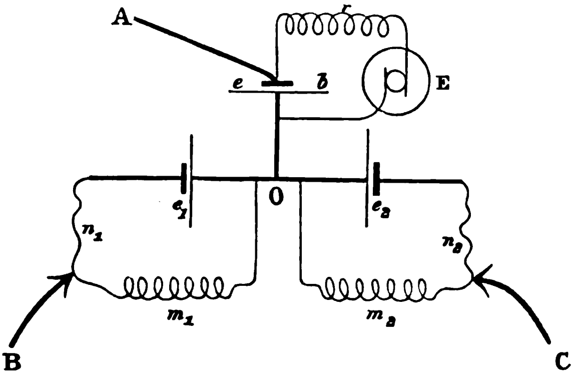
$$\therefore\ \mathrm{V}=\mathrm{C}\left(\frac{br}{b+r}+\frac{m_1n_1}{m_1+n_1}\right).$$

If we measure the apparent distance R_1 between A and B by any convenient method, and if C be the current sent through this circuit between A and B by the testing arrangement, we get R_1 equal to $\frac{\mathrm{V}}{\mathrm{C}}$;

$$\therefore\ \mathrm{R}_1=\frac{br}{b+r}+\frac{m_1n_1}{m_1+n_1}.$$

Now if we have three circuits (fig. 2) connected together

Fig. 2.



at the point O, the cell of E.M.F. equal to e and resistance b being the particular one whose resistance is required, and if the resistances m_1, m_2, n_1, n_2 be so adjusted that the points A, B, and C are at the same potential, the apparent resistances R_1, R_2, R_3 between the points A and B, A and C, and B and C, will have the following values :—

$$R_1 = \frac{br}{b+r} + \frac{m_1 n_1}{m_1 + n_1},$$

$$R_2 = \frac{br}{b+r} + \frac{m_2 n_2}{m_2 + n_2},$$

$$R_3 = \frac{m_1 n_1}{m_1 + n_1} + \frac{m_2 n_2}{m_2 + n_2};$$

$$\therefore \frac{br}{b+r} = \frac{R_1 + R_2 - R_3}{2}, \text{ say, equals } x,$$

then the required resistance of the cell,

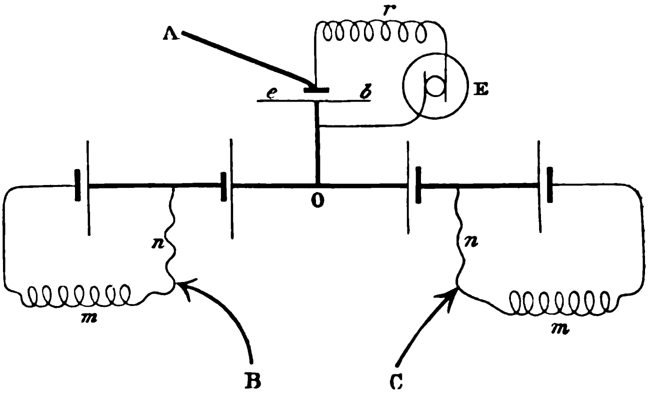
$$b = x + \frac{x^2}{r} + \frac{x^3}{r^2} + \&c.$$

If, as in the case of an accumulator, x is small compared with r, then

$$b = x + \frac{x^2}{r}.$$

When an accumulator is discharging, taking $b = x$ gives us a a value for b about 2 per cent. too low.

Fig. 2 *a*.



If the P.D. at terminals of the cell under test is greater than the E.M.F. of a single balancing cell (as is the case during charge) then the circuit must be modified as shown (fig. 2 *a*).

If a Wheatstone's bridge be employed to measure R_1, R_2, and R_3, there will be no necessity to employ any special instrument in the testing for equality of potential of the points A, B, and C. For all that need be done is to remove an infinity plug on the bridge, close the galvanometer circuit, but not that of the testing battery, and adjust m_1, m_2, n_1, n_2 until the galvanometer remains at zero.

XL. *Breath Figures.*

By W. B. Croft, *M.A.*[*]

FIFTY years back Prof. Karsten, of Berlin, placed a coin upon glass, and by electrifying it made a latent impression, which revealed itself when breathed upon. About the same time Mr. W. R. (now Sir W. R.) Grove made similar impressions with simple paper devices, and fixed them so as to be always visible. A discussion of Karsten's results occurs in several places, but I have not been able to find details of his method of performing the experiment. During my attempts to repeat it some effects have appeared which seem to be new and worthy of record.

After many trials I found the following method the most successful:—A glass plate, 6 inches square, is put on the table for insulation: in the middle lies a coin with a strip of tinfoil going from it to the edge of the glass: on this coin lies the glass to be impressed, 4 or 5 inches square, and above it a second coin. It is essential to polish the glass scrupulously clean and dry with a leather: the coins may be used just as they usually are, or chemically cleansed, it makes no difference. The tinfoil and the upper coin are connected to the poles of a Wimshurst machine which gives 3 or 4 inch sparks. The handle is turned for two minutes, during which one-inch sparks must be kept passing at the poles of the machine. On taking up the glass one can detect no change with the eye or the microscope; but when either side is breathed upon, a clear frosted picture appears of that side of the coin which had faced it: even a sculptor's mark beneath the head may be read. For convenience those parts where the breath seems to adhere will be called white, the other parts black. In this experiment the more projecting parts of the coin have a black counterpart, but there is a fine gradation of shade to correspond with the depth of cutting in the device: the soft undulations of the head and neck are delicately reproduced.

---

[*] Read June 24, 1892.

The microscope shows that moisture is really deposited over the whole surface, the size of the minute water granulation increasing as the point of the picture is darker in shade.

There seems to be no change produced by the use of coins of different metals.

If sparking is allowed across the glass instead of at the poles of the machine, traces of metal are sometimes deposited beyond the disk of the coin, but not within it.

Around the disk is a black ring ¼ inch broad: sometimes the milling of the coin causes radial lines across this halo.

If carefully protected there appears to be no limit to the permanence of the figures, but commonly they are gradually obscured by the dust gathered up after being often breathed upon: some of the early ones, done more than two years back, are still clear and well defined in the detail.

It is possible to efface them with some difficulty by rubbing with a leather whilst the glass is moist. They are best preserved by laying several together when dry and wrapping them in paper: they are not blurred by this contact.

It is a curious fact that certain developments take place after a lapse of some weeks or months. The dark ring around the disk gradually changes into a series of three or four, black and white alternately; other instances of such a change will be noted below.

Let it be noticed that in coin pictures the object is near to, but not in contact with, the glass: for in the best specimens the rim of the coin keeps the inner part clear of the surface.

Obviously a small condenser is made by the coins: it is not essential; at the same time images made by a single coin, put to a single pole, are inferior.

The plan which gives the surest and most beautiful results is to place five or six coins, lying in contact side by side in a cross or star, on either side of the glass: it is not necessary that each coin should exactly face one on the other side.

There has not appeared any distinction between the figures made by positive and negative electricity.

When several coins are side by side, touching one another, there appear in the spaces between them, which are mostly black, well-defined white lines, common tangents to the circular edges of the coins. If these are of equal size the lines are straight; otherwise they are curved, concave towards a smaller coin. They seem to be traces in that plane of the loci of intersection of equipotential surfaces.

Similar effects are obtained when coins and glasses are piled up alternately, and the outer coins are put to the poles of the machine. With six glasses and seven coins perfect images have been formed on both sides of each glass. With eight glasses the figures were imperfect; but there is little doubt this could be improved by continued trials as to the amount of electricity applied.

If several glasses are superposed and coins are applied to the outer surfaces, there are only the two images at the outside. After the electrification there is a strong cohesion between the plates.

It requires some practice to manage the electrification so as to produce the best results. There are two forms of failure which present interesting features. Sometimes a picture comes out with the outlines dotted instead of being continuous. At other times, if the electrification is carried too far, the impression comes out wholly black; but on rubbing the glass when dry with a leather the excess is somehow removed. Naturally it is difficult to rub down exactly to the right point, but I have succeeded on several occasions in developing from a blank all the fine detail of elaborate coins.

Here, again, we have another instance of the development by lapse of time, for an over-excited piece of glass usually gives a clear picture after an interval of a day or two.

Impressions from stereotype plates have been taken of which the greater part is legible: the distinctness usually improves after a few days. In default of a second plate, a piece of tin-foil about the same size should be put on the opposite side of the glass.

Sheet and plate glass of various thicknesses have been used without any noticeable change either in the treatment or the results.

I have put an impressed glass on a photographic plate in the dark, but did not get any result on developing: my imperfect skill in photographic matters leaves this experiment inconclusive.

Probably all polished surfaces may be similarly affected: a plate of quartz gives the most perfect images, which retain their freshness longer than those on glass.

Mica and gelatine give poorer results: it is not possible to polish the surface to the necessary point without scratching it.

On metal surfaces fairly good impressions can be produced if, as Karsten advises, oiled paper is put between the coin and the surface.

In the order of original discovery the figures noticed by Peter Riess should come first. He discusses a breath-track made on glass by a feeble electrical discharge; as well as two permanent marks, noticed by Ettrick, which betray a disintegration of the surface.

I have found that when a stronger discharge is employed more complex phenomena of a similar kind are produced. A 6-inch Wimshurst machine is arranged with extra condensers, as if to pierce a piece of glass. If this is about 4 inches square the spark will generally go round it. For a day, more or less, there is only a bleared watery track, $\frac{3}{10}$ inch wide, when the glass is breathed upon; but after this time others develop themselves within the first, a fine central black line with two white and two black on either side, the total breadth being the original $\frac{3}{10}$ inch. These breath-lines do not precisely coincide in position with the permanent scars, but the central one is almost the same as a permanent mark, which the microscope shows to be the surface of glass fractured into small squares of considerable regularity: on either side is a grey-blue line always visible, which Riess ascribes to the separation of the potash. After several months I found two blue lines on either side, which I believe were not visible at first. Of course these blue lines may be seen on most Leyden jars, where they have discharged themselves across the glass.

In 1842 Möser, of Königsberg, produced figures on polished surfaces by placing bodies with unequal surfaces near to them: the action was ascribed to the power of light, and his results were compared with those of Daguerre. Möser says, "We cannot therefore doubt that light acts uniformly on all bodies, and that, moreover, all bodies will depict themselves on others, and it only depends on extraneous circumstances whether or not the images become visible." In general, the multitude of images would make confusion; it can only be freshly polished surfaces that are free to reveal single definite impressions. However great Möser's assumption may be, there are many achievements of modern photography that would be as surprising if they were not so familiar. I have not the means of knowing the precise form of Möser's methods: in the experiments which follow there is usually contact and light pressure, and if they are not wholly analogous, they may for that cause help to generalize the idea: in none of these is electricity applied.

A piece of mica is freshly split, and a coin lightly pressed for 30 seconds on the new surface: a breath-image of the coin is left behind. At the same time it may be noticed that the breath causes abundant iridescence over the surface, whilst it is in a fresh state. It is not clear how the electricity of cleavage can have an active agency in the result.

It is familiar to most people that a coin resting for a while on glass will give an outline of the disk, and sometimes faint traces of the inner detail when breathed upon.

An examination-paper, printed on one side, is put between two plates of glass and left for ten hours, either in the dark or the daylight: a small weight will keep the paper in continuous contact, but this is not necessary if thick glass is used. A perfect breath-impression of the print is made, not only on the glass which lay against the print but also on that which faced the blank side of the paper. Of course the latter reads directly, and the former inversely; the print was about one year old, and presumably dry.

More often both impressions are white, sometimes one or other or both are black. At other times the same one may be part white and part black, and they even change while being examined.

During a sharp frost with east winds early in March, 1890, these impressions of all kinds were easy to produce, so as to be quite perfect to the last comma; but in general they are difficult, more especially those from the blank side.

At the best period those from the blank side of the paper were white and very strong; also there were white spots and blotches revealed by the breath. They seemed to correspond with slight variations in the structure of the paper, and suggest an idea that the thickness of the ink or paper makes a minute mechanical indentation on the molecules: the state of these is probably tender and sensitive under certain atmospheric conditions, as happens with steel in times of frost.

The following experiments easily succeed at any time:—Stars and crosses of paper are placed for a few hours beneath a plate of glass: clear white breath-figures of the device will appear. A piece of paper is folded several times each way to form small squares, then spread out and placed under glass: the raised lines of the folds produce white breath-traces, and a letter weight that was above leaves a latent mark of its circular rim.

Some writing is made on paper with ordinary ink and well dried: it will leave a very lasting white breath-image after a few hours' contact. If, with an ivory point, the writing is traced with slight pressure on glass, a black breath-image is made at once. Of course this reads directly, and the white one inversely. It is convenient to look through the glass from the other side for inverse impressions, so as to make them read direct.

Plates of glass lie for a few hours on a table-cover worked with sunflowers in silk: they acquire strong white figures from the silk.

In most cases I have warmed the glass, primarily for the sake of cleansing it from moisture; but I have often gone to a heat beyond what this needs, and think that the sensitiveness has been increased thereby.

It is not not easy to imagine what leads to the distinction between black and white, different substances act variously in this respect. I have placed various threads for a few hours under a piece of glass, which lay on them with light pressure: wool gives black, silk white, cotton black, copper white. A twist of tinsel and wool gives a line dotted white and black; after a time these traces show signs of developing into multiple lines as in the spark figures.

Two cases have been reported to me where blinds with embossed letters have left a latent image on the window near which they lay; it was revealed in misty weather, and had not been removed by washing. I have not had a chance to see these for myself, but both my informants were accustomed to scientific observation.

A glass which has lain above a picture for some years, but is kept from contact by the mount, will often show on its inner side an outline of the picture, always visible without breath. It seems to be a dust figure easily removed: possibly, heat and light have loosened fine paint particles, and these have been drawn up to the glass by the electricity made in rubbing the outer side to clean it. The picture must have been well framed and sealed from external influences; most commonly dust and damp get in and obscure such a delicate effect.

I am not able to suggest simple causes for these varied effects. I am not inclined to think, except in the case of water-colours, which is hardly part of the inquiry, that there is a definite material deposit or chemical change; one cannot suppose that imperceptible traces of grease, ineradicable as they may be, would produce complete and delicate outlines. The cleaning off of impressions may at first seem to indicate a deposit; but this renewal of the surface might rather be like smoothing out an indented tin-foil surface: such a view might explain the case where a blank over-electrified disk is developed into fine detail. The electrified figures seem to point to a bombardment, which produces a molecular change, the intensity of electricity bringing about quickly what may also be done by slow persistent action of mechanical pressure. At present it seems as if most of the phenomena cannot be drawn out from the unknown region of molecular agency.

While experimenting I was not within reach of references to former researches, but I have since done my best to find them out, and to indicate all I have learnt in the body of my paper.

Poggendorff, vol. lvii. p. 492; translated in *Archives de l'Electricité*, 1842, p. 647.
Riess' *Electrische Hauchfiguren* in *Repertorium der Physik*; translated in *Archives de l'Electricité*, 1842, p. 591.
Riess' *Die Lehre von der Reibungs Electricität*, vol. ii. pp. 221–224.
Mascart, *Electricité Statique*, vol. ii. p. 177.
Taylor's Scientific Memoirs, vol. iii.

XLI. *Inductoscript.* *By* FREDERICK J. SMITH, *M.A., Millard Lecturer Mech. et Phys., Trinity College, Oxford* *.

THE well-known breath figures of Möser, Riess, and Karsten suggested to me the idea of producing similar figures on photographic plates; my laboratory day-book for 1878 shows that I tried an experiment to determine the action of a discharge from a coin on to a photo-plate. The image was produced on development, but very imperfectly. During the last two years an inductorium or transformer has been under constant trial in my laboratory. It occurred to me that probably its high potential might be used to produce figures similar to those of Karsten: plates of glass were cleaned with care, and operated on in the way suggested by Karsten, but no results were to be got out of it; probably the potential difference was not great enough.

I then went on to place a photo-plate in the place of the glass, and on development the exact pattern of the medal used was produced; this showed that a PD, greatly under that of a jar discharge, or that of an electrical machine, would upset the chemical equilibrium of the photo-plate. Several factors appeared to be contributing to the result—the potential difference; the gas in which the experiment was performed; the pressure under which the experiment was done; the temperature; and the history of the plate previous to the experiment.

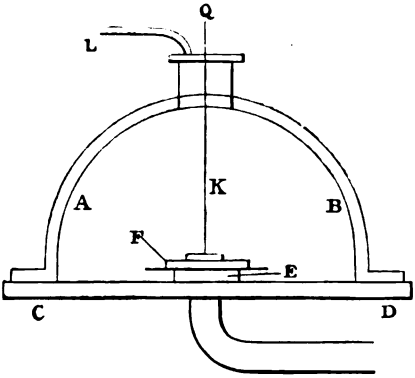
---

* Read June 24, 1892.

The following inductoscripts (a name I venture to propose, as it somewhat suggests the nature of the process, without being hybrid in construction), Nos. *A* to *W*, were produced under varying circumstances.

The experiments were performed under a bell-jar, A, B, fig. 1. This was placed on the air-pump plate, C, D; a round copper disc, E, was supported on the pump-plate on three points; on this disc a sensitive plate, F, was placed, and on this the coin the print of which was to be produced. The electricity was introduced through the copper rod K, and the plate and rod were connected to the terminals of the inductorium, driven by four accumulators in series, each having seven plates 12″ × 12″.

Fig. 1.



By means of the curved tube, L, different gases were introduced into the bell-jar.

The intensity of the spark could be regulated by bringing the terminals of the inductorium together; when they were brought to within 4 mm. of each other the discharge was called *slight*, when placed wide apart it was called *full on* or *full*.

The following results were obtained :—

No.	Time of exposure (seconds).	Gas.	Electricity.	Pressure in receiver.
A.	10	Oxygen	+ Pos. Slight	760
B.	10	Air	+ Slight	760
C.	10	Air	+ Full	760
D.	5	Oxygen	+ Moderate	760

No.	Time of exposure (seconds).	Gas.	Electricity.	Pressure in receiver.
E.	10	Oxygen +	Moderate	760
F.	10	Oxygen +	Moderate	760
G.	30	Oxygen +	Slight	760
H.	5	Coal-gas +	Full	760
I.	20	Coal-gas −	Slight	760
J.	50	Air −	Neg.	760
K.	5	Air −	Full	760
L.	20	Air −	Slight	760
M.	5	Air −	Slight	760
N.	2	Air −	Slight	760
O.	5	Oxygen −	Moderate	760
P.	5	Air −	Moderate	760

Effects of a vacuum :—

No.	Time (seconds).	Gas.	Pressure (indicated by barometer-tube on pump).
Q.	5	Air	745 mm.
R.	5	Air	745 mm.
S.	5	Air	755 mm., point of conductor 2·5 cm. above coin, ½ sovereign.
T.	5	Air	630 mm., point ½ cm. above coin.
U.	5	Air	755·5 mm., point ½ cm. above coin.
V.	5	Oxygen	700, point ½ cm. above coin.
W.	10	Air	760, plate exposed to daylight before being used, the image developed out, but the whole plate quickly turned black all over.
X.	10	Oxygen	760.

It will be noticed that in *V* and *X* the discharge has produced the peculiar effect of transparent streaks in one direction, and black opaque streaks in an opposite direction. During the development the black streaks came out first, and some time afterwards the transparent ones. Experiments bearing on this point will be repeated.

It is evident from the inductographs taken *in vacuo* that a good vacuum materially checks, and nearly stops, the phenomenon taking place.

It is to be noticed in reading the work on breath figures by Grove and Karsten, that Grove tried to set the image by coating an electrised plate with collodion and nitrate of silver, and then developed out the word Volta (sec. 4, p. 404, 'Correlation of Physical Forces,' Grove). He did not let *the electrical discharge act on the silver salt.* Again, Karsten did not succeed in producing a fixed picture on a daguerreotype plate; he writes, "If an iodized plate is taken, and a picture produced thereon, the vapour of mercury will not bring it out, although the image is really on the plate, as may be shown by breathing upon it." Möser appears to have got out pictures by means of water-vapour, but they were not permanent effects.

As a summary of my own work, I would mention that—

I. The different gases used were oxygen, carbonic acid gas, air, coal-gas. Far the best results were obtained in oxygen. Experiments with other gases are now being made.

II. The effect of pressure. No picture could be obtained in a good vacuum, only the circular shape of the coin; as as the pressure increased towards the normal the pictures became more perfect.

III. The output of the transformer was regulated by resistances put into the primary circuit, also by the rate at which the reversing commutator was driven; the reversing commutator is similar to those used on the early Siemens machine, and driven from a belt in the laboratory.

IV. The temperature varied between 5° C. and 100° C. Perhaps the effects may have been produced more quickly at the higher temperature, but no significant change was apparent.

V. Similar experiments are about to be performed in many other gases, and under considerable pressure, up to 10 atmos. The apparatus is nearly ready.

VI. In addition to pictures on photo-plates, good impressions have been obtained on bromide paper and other papers direct. Pictures can easily be got from woodcuts, after they have been well covered with plumbago.

XLII. *On the Relation of the Dimensions of Physical Quantities to Directions in Space* [*]. *By* W. WILLIAMS, *Assistant in the Physical Laboratory, Royal College of Science* [†].

IN a paper read before the Physical Society, Nov. 24, 1888, Prof. Rücker showed that in the dimensional formulæ of electromagnetic quanties μ and k the two specific capacities of the medium are generally omitted, or rather their dimensions are suppressed, and that the consequence of this suppression is to give rise to two artificial systems of dimensions, the electrostatic and the electromagnetic. That these systems are artificial appears when we consider that each, apparently, expresses the *absolute* dimensions of the different quantities, that is their dimensions only in terms of L, M, and T; whereas we should expect that the absolute dimensions of a physical quantity could be expressed only in one way. Thus, from the mechanical force between two poles, we get

$$f=\frac{1}{\mu}\left(\frac{m^2}{r^2}\right), \qquad \therefore \quad m=r\sqrt{\mu f};$$

and this, being a qualitative as well as a quantitative relation, involves the dimensional identity of the two sides. If now μ is put $=1$, we either *ignore* its dimensions or *assume* that it has none, being but a mere number. In the former case the dimensions deduced for m under such circumstances must be admittedly artificial, since if the suppressed dimensions of μ were inserted, those of m would be different, and would involve a different physical interpretation. In the latter case the dimensions deduced for m must be its absolute dimensions and must therefore involve its physical interpretation. But if we start with the relation

$$f=\frac{1}{k}\left(\frac{q^2}{r^2}\right), \qquad \therefore \quad q=r\sqrt{kf},$$

where f is now the force between two charges q, and if we treat k as a pure number, we obtain for m by means of the dimensions of q deduced by this relation absolute dimensions again, but different now to what we had before. In this way we get two different absolute dimensions for the same physical quantity,—each of which involves a different physical interpretation. If, however, μ and k be not mere numbers, but concretes expressing physical properties of the medium, as is indeed proved by the relation

---

[*] See Note at the end of the Paper.

[†] Read June 24, 1892, and communicated by A. W. Rücker, F.R.S.

$$[\mu k] = (\mathrm{L}^{-2}\mathrm{T}^{2})^{*},$$

then the fact that the dimensions of electrical and magnetic quantities are different in the two systems arises only from our ignorance of the dimensions of μ and k, which must be such as to bring the two systems into accord when ultimately expressed in terms of L, M, and T. Prof. Rücker therefore suggested that μ and k should be admitted into the formulæ along with L, M, and T, and that, since nothing is definitely known as to their physical nature, they should be regarded as secondary fundamental units, additional to L, M, and T, but which must be ultimately expressed in terms of them. There is thus, in each formula, an indication that the irrational and unsuggestive dimensions are not absolute as they stand; each formula is made to express a physical reality, and each contains in the proper form the factor necessary to render the formula *absolute.* This system has been adopted, among others, by Prof. Gray in his smaller edition of 'Absolute Measurements in Electricity and Magnetism,' and by Prof. Everett in his last edition of 'Units and Physical Constants.'

Prof. Fitzgerald has pointed out (Phil. Mag. April 1889) that if a system be taken "in which the dimensions of μ and k are the same, and of the dimensions of a slowness, that is the inverse of a velocity $(\mathrm{L}^{-1}\mathrm{T})$, then the two systems become identical as regards dimensions, and differ only by a numerical coefficient just as centimetres and kilometres do." But although μ and k are quantities of the same order, both being capacities, their dimensions need not necessarily be the same, unless electrification and magnetization be phenomena of the same order as well. If, however, these be different orders—as they almost certainly are—the one possibly a strain, the other a vortex motion, then it is unlikely that k and μ, bearing, as each does, similar relations to two dissimilar phenomena, should have identical dimensions.

---

* See also "On the Dimensions of a Magnetic Pole in the Electrostatic System of Units," Dr. Lodge, Phil. Mag. Nov. 1882.

In one of the Appendices to 'Modern Views of Electricity' Dr. Lodge develops a system in which μ and k^{-1} have respectively the dimensions of *density* and *rigidity*. It is then found that the dimensions of all the other quantities become unique, and capable of purely dynamical interpretations. Thus, magnetic moment becomes linear momentum; magnetic induction, linear momentum per unit volume; magnetic force, velocity; electrical force, pressure ; current, displacement × velocity; self-induction, "inertia per unit area," &c. In a similar manner other such systems may be developed. Thus, take the equation

$$F = M\frac{\partial v}{\partial t} + kv,$$

where F is an impressed force, M the mass of a moving body, v its velocity, and k a coefficient of resistance. Compare this with

$$E = L\frac{\partial C}{\partial t} + RC,$$

where E is the voltage of a closed circuit, L its self-induction, C the current, and R the resistance. Let us identify E and F dimensionally, and work out the analogy in detail. To do this, we have only to equate the dimensions of E and F, and find what value of μ satisfies the relation. Then substituting this value of μ in the formulæ of the other quantities, we are able to complete a connected dynamical analogy of electromagnetism by starting with *voltage* as a *force*. In this case, we find that electrification is a displacement ; current, a velocity ; electrical potential, force ; quantity of magnetism, linear momentum; self-induction, inertia, &c. In a precisely similar manner, by starting with the equation

$$G = I\frac{\partial \omega}{\partial t} + B\omega,$$

where G is a couple, I a Moment of Inertia, and ω an angular velocity, we get:—electrification, a strain; electrical potential, work; electrical force, force; current, angular velocity; quantity of magnetism, angular momentum; self-induction, moment of inertia, &c. In all cases we are able, by means of the dimensional formulæ expressed in terms of μ and k, to work in detail any dynamical analogue we may choose to take as a starting-point. Of course all dimensional values of μ, together with the corresponding ones for k, must render electromagnetic dimensions unique. It is only some of these, however, that give rise to formulæ capable of dynamical interpretation, and it will be found on examination that the number of such values is small. Now, if electromagnetism is ultimately dynamical, the dimensions of electromagnetic quantities must be of the same kind (ultimately) as those of ordinary dynamical units. Hence, by examining all the possible cases in which the dimensions of μ and k lead in the case of the other electro-magnetic quantities to dimensions of the dynamical order, we *may* be able to obtain some light as to the nature of μ and k themselves. To show how this may be done, and the kind of results obtained will be the object of the following paper.

Before doing this, however, it becomes necessary to examine in more detail the real nature of dimensional equations, and to determine how far they are capable of giving reliable results when used in the way here suggested. Primarily, a dimensional formula expresses only numerical relations between units, and for the purpose of the present paper is defective from the fact that different physical quantities have the same dimensional formulæ. For example, *couple* and *work*, as pointed out by Prof. S. P. Thompson in the course of the discussion on Prof. Rücker's paper above referred to; or, again, an *area* and the *square of the same vector-length*; *pressure* and *tangential force per unit area*, &c. If, therefore, dimensional equations are to be used at all in the sense above indicated, it becomes necessary, in the first place, to be assured that the process is valid, and, in the second place, that no contradictory or unintelligible results arise from causes such as the above.

The dimensional formula of a physical quantity expresses the *numerical* dependence of the unit of that quantity upon the fundamental and secondary units from which it is derived, and the indices of the various units in the formula are termed the *dimensions* of the quantity with respect to those units. When used in this very restricted sense, the formulæ only indicate numerical relations between the various units. It is possible, however, to regard the matter from a wider point of view, as has been emphasized by Prof. Rücker in the paper above referred to. The dimensional formulæ may be taken as representing the *physical identities* of the various quantities, as indicating, in fact, how our conceptions of their physical nature (in terms, of course, of other and more fundamental conceptions) are formed—just as the formula of a chemical compound indicates its composition and chemical identity. This is evidently a more comprehensive and fundamental view of the matter, and from this point of view the primitive numerical signification of a dimensional formula as merely a change ratio between units becomes quite a dependent and secondary consideration.

The question then arises, what is the test of the identity of a physical quantity? Plainly, it is the manner in which the unit of that quantity is built up (ultimately) from the fundamental units L, M, and T, and not merely the manner in which its magnitude *changes* with those units. Thus, the unit couple and the unit of work both change in the same manner with the unit length, but they are physical quantities of different kinds. Their *numerical* dependence upon L, M, and T is the same, as expressed by the formula $(MLT^{-2})L$, but the manner the unit length enters their definition is different: in the case of work the two units of length involved are in the same direction; in the case of couple they are mutually at right angles. The absolute measure of a force is the work done through unit linear displacement; similarly, the absolute measure of a couple is the work done through unit angular displacement. Hence the relation between couple and work is similar to that between force and work, the difference being that angular displacements are considered in the former case, linear displacements in the latter. But the measure of an angular displacement is independent of the unit length. Hence, in expressing the numerical dependence of the unit couple and the unit of work upon the fundamental units we get the same formula ML^2T^{-2}, although the difference in the physical nature of the quantities is of the same kind as that between force and work. And generally, when used in the purely numerical sense above indicated, the dimensional formulæ fulfil all requirements; it is only when endowed with the higher function of defining the physical identities of the various quantities that they are found to fail.

That the dimensional formulæ *are* regarded from this higher standpoint—that is, regarded as being something more than mere "change ratios" between units—is shown by the fact that difficulties are felt when the dimensions of two different quantities, *e. g.*, couple and work, happen to become the same. If, however, the numerical dependence of the units of the two quantities upon the fundamental units be the same, and if the formulæ are to express nothing more, then the two quantities *must* have the same dimensions, and from this point of view we are not entitled to feel any difficulty in the matter. That such difficulties are felt arises therefore from the more comprehensive signification which is attached to the formulæ—a signification which obviously includes all the numerical considerations which alone constitute the more restricted one.

Let us, therefore, for the purpose of the present paper, regard the dimensional formula of a quantity as the symbolical expression of the physical nature of that quantity, so far, of course, as it depends upon the fundamental conceptions of mass, space, and time (and, in the case of thermal and electrical quantities, of secondary conceptions also ultimately dependent upon mass, space, and time). To obtain the formula for any quantity, it is only necessary to express how the unit of the quantity is built up from the fundamental and secondary units. Now, the units of mass and time, and all secondary units, are involved in all physical units in a simple manner. They are raised to different powers. But, owing to the dimensions of space, the unit of length is involved in different ways, according to the different relative directions in which it may be taken. In all cases, however, the unit of any quantity can be completely expressed—so far as it involves the unit length—by taking the unit length along one or more of three mutually rectangular directions Ox, Oy, Oz, whose absolute direction in space is of course determined by the nature of the physical relation into which the quantity enters. The dimensional formulæ can therefore be expressed in terms of M, T, and X, Y, Z, where M is the unit of mass, T the unit of time, and X, Y, Z the unit of length taken respectively along the directions Ox, Oy, Oz. Thus, if MXT^{-2} is the unit of force, MX^2T^{-2} is the unit of work ; $MXY^{-1}Z^{-1}T^{-2}$, energy per unit volume ; $MXZT^{-2}$, a couple in the plane XZ, &c. The above quantities are, of course, the same in kind but different in direction from MYT^{-2}, MZT^{-2}, &c. This method of expressing the dimensional formulæ is nothing more than a modified extension of Prof. S. P. Thompson's suggestion as to the use of $\sqrt{-1}$ in dimensional formulæ, as will appear more fully later (see discussion on Prof. Rücker's paper above referred to).

When it is necessary to determine the *numerical* dependence of the unit of a quantity upon the fundamental and secondary units, the distinction between units of length taken in different directions must be suppressed, and the unit of length in whatever direction taken must be represented by a single symbol L. The index of L will therefore be the *sum* of the indices of X, Y, and Z, and the formulæ thus simplified will indicate the changes in the magnitude of the unit when the fundamental and secondary units are themselves changed. Thus, we can immediately deduce from any formula the particular and simplified form it assumes when only the numerical relations between units are to be considered. It will be convenient to designate these simplified forms of the formulæ the "change ratios" of the units, and to reserve the term "dimensional formulæ" for the more general forms in which the identities of the quantities are primarily concerned.

The dimensions of the ordinary dynamical quantities may now be expressed :—

1. Inertia = M.
2. Length = X, Y, or Z.
3. Area = XY, YZ, or ZX.
4. Volume = XYZ.
5. Density = $MX^{-1}Y^{-1}Z^{-1}$.

6. Velocity $=XT^{-1}$, YT^{-1}, ZT^{-1}.
7. Acceleration $=XT^{-2}$, YT^{-2}, ZT^{-2}.
8. Linear momentum $=MXT^{-1}$, MYT^{-1}, MZT^{-1}.
9. Force $=MXT^{-2}$, MYT^{-2}, MZT^{-2}.
10. Work $=MX^2T^{-2}$, MY^2T^{-2}, MZ^2T^{-2}.
11. Energy per unit volume $=MXY^{-1}Z^{-1}T^{-2}$, $MX^{-1}YZ^{-1}T^{-2}$, $MX^{-1}Y^{-1}ZT^{-2}$.
12. Pressure $=MXY^{-1}Z^{-1}T^{-2}$, [Force $=MXT^{-2}$, per unit of surface YZ] &c.

Thus, the dimensions of pressure are the same as those of energy per unit volume, as should be the case, for the pressure at a point in a gas is given by $p=\frac{1}{3}\rho\overline{V}^2$, where ρ is the density and $\rho\overline{V}^2$ is of the dimensions of energy per unit volume. The case is different from that of $W=G\theta$, where W is the work done by a couple G through an angular displacement θ. In the former case, p is the same in *kind* as $\rho\overline{V}^2$, nothing concrete intervening. In the latter case, G cannot be the same as W unless θ be a mere number, having no reference to anything concrete, which is not the case.

13. Couple $=MXYT^{-2}$, (Force along X *or* Y, Arm Y *or* X) &c.

The rational measure of an angle is $\frac{s}{r}$, where s is the arc described by the radius r rotating about one extremity. Let ∂s be an element of the arc, then $s=\Sigma\partial s$, and $\theta=\frac{\Sigma\partial s}{r}$, or $\partial\theta=\frac{\partial s}{r}$. Let r be measured along X, then, since ∂s is always an element at right angles to r, ∂s will be measured along Y or Z. If these directions (X, Y, Z) be carried along *with* r—that is, if we take instantaneous axes at every point of the arc—the axes bearing the same relation to the radius and tangent at every point, we get $\partial\theta=\frac{\partial y}{x}$, and $\theta=\frac{\Sigma\partial y}{x}$. To express this dimensionally, we have to neglect the summation Σ; for a dimensional formula expresses the *nature* of a quantity, not its magnitude, and the same formula must therefore apply to both θ and $\partial\theta$. The dimensions of θ and $\partial\theta$ are therefore YX^{-1}.

14. Angles $= XY^{-1}$, (Direction of radius Y, of the arc X) &c.
15. Angular Velocity $= XY^{-1}T^{-1}$, &c.
16. Angular Acceleration $= XY^{-1}T^{-2}$.
17. Moment of Inertia $= MX^2, MY^2, MZ^2$.

$$\frac{\text{Moment of couple}}{\text{Angular acceler.}} = \frac{MXYT^{-2}}{XY^{-1}T^{-2}} = MY^2, \text{\&c.}$$

18. Angular Momentum $= I\omega = MY^2(XY^{-1})T^{-1} = MXYT^{-1}$, &c.
19. Energy of Rotation $= \frac{1}{2}I\omega^2 = MY^2(XY^{-1})^2T^{-2} = MX^2T^{-2}$, &c.
20. Couple $= I\frac{\partial\omega}{\partial t} = MY^2(XY^{-1})T^{-2} = MXYT^{-2}$.
21. Work done by a couple $= G\theta = (MXYT^{-2})(XY^{-1}) = MX^2T^{-2}$.
22. Solid angle $= \partial\Omega = \frac{\partial a}{r^2}$, where ∂a is an element of area on the surface of a sphere radius r. Taking instantaneous axes, X being always along the radius, and Y, Z in the tangent plane at a point, this becomes dimensionally,

$$[\partial\Omega] = (YZX^{-2}).$$

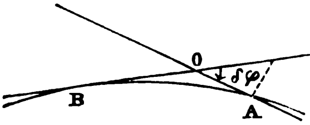
The quantity π may enter into physical relations in two different ways. It may enter in its purely geometrical or trigonometrical capacity as a definite number of radians (or, what is equivalent, as the ratio of the circumference of a circle to its diameter), and be thus definitely related in a physical sense to the other quantities ; or it may enter in its numerical capacity as part of a numerical coefficient determined by higher abstract analysis. Cases of the former kind only have to be considered in the following paper. In these cases π enters the relations (ultimately) as a definite number of plane or solid angles, and it therefore consists of two factors, namely, a numerical factor, 3·14···, and a concrete factor, the unit plane or solid angle. We may therefore speak of the dimensions of π, meaning thereby the dimensions of the plane or solid angles which it *in such cases* implies, and the dimensional identity (in the directional sense) of the relations into which it so enters cannot be complete if the concrete factor is neglected. Thus, let $A=4\pi r^2$, where A is the surface of a sphere of radius r. Taking axes as in 22, this becomes dimensionally $YZ=(YZX^{-2})X^2$, and the relation is not true if 4π be treated as a pure number unless an area is represented by the square of a vector length, instead of by the product of *two* different vector lengths as in vector algebra. Similarly, in the relation $V=\frac{4}{3}\pi r^3$, V being a volume, π is of the dimensions of a solid angle, while in $l=2\pi r$, where l is the circumference of a circle, it is of the dimensions of a plane angle.

The significance of π as it occurs in electromagnetism in connexion with radial and circuital fluxes will be considered later.

The following examples serve to illustrate this method of expressing dimensional formulæ.

23. *Radius of Curvature.*—Bending per unit length of a curve. Let A B be an element of a curve, and let the tangents at A and B intersect at O, making an angle $\partial\phi$. The angle $\partial\phi$ is ultimately $=\frac{AP}{OP}$. If A P be taken along the normal X at A, and O A along the tangent Y, then A B is also ultimately along Y, and may be written ∂y. Hence the curvature becomes $\frac{\partial\phi}{\partial y}$, or dimensionally $(XY^{-1})Y^{-1}=XY^{-2}$, where XY^{-1} are the dimensions of $\partial\phi$, and Y those of AB. The radius of curvature at O is therefore of the dimensions (Y^2X^{-1}).

Fig. 1.



24. *Centrifugal Force.*—Let a particle describe the path A B with velocity V, the centrifugal force is $F=\left(m\frac{V^2}{\rho}\right)$. Taking instantaneous axes at A, as in 23, this becomes dimensionally

$$[F]=M(Y^2T^{-2})(XY^{-2})=MXT^{-2}.$$

Thus the force is directed along the radius X.

25. *Compressibility.*—Hydrostatic pressure is of the dimensions $MXT^{-2}(YZ)^{-1}$, according to the direction of the plane of reference YZ. Strain is here of no dimensions, being the ratio of two concretes of the same kind. Hence the dimensions of compressibility are $M^{-1}X^{-1}YZT^{2}$.

26. *Rigidity.*—"A simple shear is a homogeneous strain in which all planes parallel to a fixed plane are displaced in the same direction parallel to that plane and therefore through spaces proportional to their distances from that plane" (Kelland and Tait's 'Quaternions,' p. 204). Let the planes of displacement be the planes XY, and the direction of displacement X. The applied stress (tangential force per unit area) is

$$\frac{MXT^{-2}}{XY}=MY^{-1}T^{-2}.$$

The shear is XZ^{-1}. Hence the rigidity is of the dimensions

$$\frac{MY^{-1}T^{-2}}{XZ^{-1}}=(MX^{-1}Y^{-1}ZT^{-2}).$$

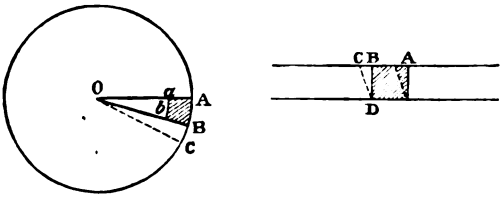
The velocity of propagation of a wave of distortion is $V=\sqrt{\frac{n}{\rho}}$, $n=$ rigidity. Hence

$$[V]=\sqrt{\frac{MZ(YX)^{-1}T^{-2}}{M(XYZ)^{-1}}}=ZT^{-1}.$$

Thus, the applied stress acts over the face XY, and the disturbance is propagated along Z.

27. *Torsion.*—Let OABC be the section of a cylinder, axis Z and radius Y, subjected to torsion; *a*AB*b* the face of an elementary cube edge ∂r, bounded as in the figure; and θ the twist per

Fig. 2.



unit length. The shear experienced by the elementary cube is ultimately $\frac{BC}{BD}=r\theta$, or dimensionally XZ^{-1} (for θ is of the dimensions $XY^{-1}Z^{-1}$, and $r=Y$). Let $P=nr\theta$ be the applied stress (tangential force per unit area). Then $[P]=MY^{-1}T^{-2}$. The tangential force over ring of which $aABb$ forms part $=(2\pi r\partial r)nr\theta=2\pi r^2 n\theta\partial r$, or dimensionally $MY^{-1}T^{-2}(XY)=MXT^{-2}$, and the moment of this round $O=2\pi r^3 n\theta\partial r$, or dimensionally $MXYT^{-2}$. The moment due to the whole face

$$=2\pi n\theta\Sigma_0^R r^3\partial r=\left(\frac{\pi nR^4}{2}\right)\theta,$$

where $R=OA$, and this is of the same dimensions as $2\pi nr^3\theta\partial r$, for the r^4 in the former expression is of the same dimensions as $r^3\partial r$ in the latter. Thus $\frac{\pi nr^4\theta}{2}$ is of the dimensions $MXYT^{-2}$.

To determine from this the dimensions of π we have

$$\begin{aligned}
[\pi nr^4\theta] &= MXYT^{-2},\\
[\theta] &= XY^{-1}Z^{-1},\\
[r^4] &= Y^4,\\
[n] &= MZ(XY)^{-1}T^{-2};
\end{aligned}$$

$$\therefore\ [\pi]MZ(XY)^{-1}T^{-2}.Y^4.(XY^{-1})Z^{-1}=MXYT^{-2},$$

$$\therefore\ [\pi]=XY^{-1}.$$

Thus π in this case is of the dimensions of a plane angle in the plane XY.

28. *Viscosity.*—The viscous resistance between planes moving with relative velocity $u=\frac{\partial x}{\partial t}$ is given by $F=\frac{1}{4}\left(Nm\omega L\frac{\partial u}{\partial z}\right)$ (Clausius). Taking the components of ω and L along Z (normal to direction of motion) this becomes dimensionally

$$[F]=\left(\frac{M}{XYZ}\right)\left(\frac{Z}{T}\right)(Z)\left(\frac{X}{ZT}\right)=MY^{-1}T^{-2}=\frac{MXT^{-2}}{XY},$$

a tangential force in the plane of motion (XY). The coefficient of viscosity is

$$\eta=\frac{F}{\frac{\partial u}{dz}}=\frac{MY^{-1}T^{-2}}{XZ^{-1}T^{-1}}=MZ(YZ)^{-1}T^{-1}$$

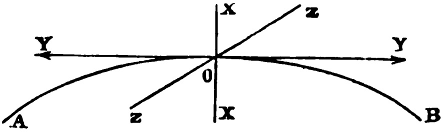
= Tangential force per unit area ÷ shear per unit time.

29. *Surface Tension.*—Tangential force per unit length normal to itself. Let Y be the direction of force in the plane

YZ, and K the surface tension. Then

$$[K]=MYZ^{-1}T^{-2}=\frac{MY^2T^{-2}}{YZ}$$
$$=\text{Energy per unit area.}$$

Fig. 3.



Let AOB be a normal section of a cylindrical liquid film, axis Z, radius X. The normal pressure due to the curvature is $K\left(\frac{1}{\rho}\right)$, or dimensionally

$$MYZ^{-1}T^{-2}(XY^{-2})=MX(YZ)^{-1}T^{-2};$$

as should be the case.

The above examples are sufficient to illustrate the method of expressing dimensional formulæ in terms of X, Y, Z. In all the above cases isolated quantities only are dealt with, and therefore the relation of the directions of directed quantities to the dimensional axes is a matter of indifference. This is not so in the case of *equations* between quantities. Here, however, we have only to transform the equations to Cartesian co-ordinates and then express the dimensions of each term by means of the above. Thus, let A, B, C, . . . be quantities connected by the equation

$$A+B+C+D+\ \dots\ =0.$$

Expressed in Cartesian co-ordinates this becomes

$$(A_x+A_y+A_z)+(B_x+B_y+B_z)+(C_x+C_y+C_z)+\ \dots\ =0;$$

and we can now immediately express the dimensions of every term. We thus get the dimensional expression of the equation itself. A physical equation implies that the quantities connected are the *same in kind*. There must, therefore, be a dimensional identity between the various terms. This holds, however, in its wider or directive sense only of component quantities in the same direction, for if we equate any term in the above to all the rest we always get a term such as A_x

equated to another such as A_y, the same in kind, but different in direction. If, however, we write the equation

$$(A_x+B_x+C_x+\ldots)+(A_y+B_y+\ldots)+(A_z+B_z+\ldots)=0,$$

there must now be a dimensional identity in the directional (X, Y, Z) sense between the terms in the same bracket.

Up to the present the dimensional formulæ have been regarded as purely conventional. It is now necessary to examine whether the conventions made use of can be justified in any other way; and it becomes important to have a clear *physical* distinction between pure numbers and concretes, for, according to the above conventions, dimensional formulæ are now extended to include quantities hitherto regarded as pure numbers.

Every concrete quantity should have definite physical dimensions in the extended sense of the term; only pure numbers should be quantities of no dimensions. For physical purposes, a pure number may be defined as the ratio of two concretes of the same kind. If the concretes are directed quantities, their ratio is not a pure number unless they are *similarly directed.* Thus, the ratio of a force along X to that along Y is of the dimensions $(MXT^{-2})/(MYT^{-2})=XY^{-1}$; that is, of the dimensions of an angle, or rotation, indicating that to compare the scalar magnitude of the two forces, they have to be rotated into coincidence. And, generally, the ratio of any two directed quantities of the same kind is a pure number together with a quantity of the nature of rotation, the latter being the *versor* part of the quotient of two vectors. Since the measure of a rotation is independent of the unit length, the versor part of the quotient is a pure number so far as the scalar unit length is concerned. It cannot, however, be truly a pure number, for different degrees of rotation are compared in terms of definite units of rotation. Hence, a *versor*, or its equivalent, an angle, or an angular displacement is a concrete quantity equally well with mass, length, and time, and should have its own proper dimensions. It is only because of the restriction of the term "concrete" to quantities the magnitudes of whose units *change* with the units of length, mass, and time that concretes of the nature of angles, and angular displacements, come to be regarded as purely numeric. From this point of view, therefore, a pure number is always of the nature of a *tensor*, or rather, unity must be regarded as the dimensional formula of a *tensor*, and of physical quantities of like nature, e. g. *volume-strains* *.

Let α, β, γ be three vectors (non-coplanar). Then

1. α, β, or γ = Linear displacements, or lengths in magnitude and direction.

2. $V(\alpha\beta)$ &c. = Area of parallelogram bounded by α, β.

3. $S(\alpha\beta\gamma)$ = Volume (neglecting sign) of parallelopiped defined by α, β, γ.

4. $U\frac{\alpha}{\beta} = \mathit{Versor}\frac{\alpha}{\beta}$ = Angular displacement required to bring β parallel to α.

If α, β, γ be vectors mutually at right angles, it is unnecessary to distinguish between scalar and vector products, and if they be also equal, between tensors and versors, for the product of any two vectors is now a vector, and of three a scalar: the quotient of any two is a versor, since the tensor $=1$; while the product of parallel vectors is always scalar. We may therefore take α, $\alpha\beta$, $\alpha\beta\gamma$, $\frac{\alpha}{\beta}$, and $\frac{\alpha}{\alpha}$ (or *unity*) as the typical representatives of the respective ideas of vector lengths, areas, volumes, angles, and pure numbers (*tensors*). If, now, we call these the dimensional formulæ (that is, the physical representations) of these ideas, we see that the tensor, although a *quasi*-physical quantity like the versor, becomes physically represented according to the above conventions by unity, while the versor is represented by the ratio of two vectors. Thus, the meaning of the fact that *tensors* and physical quantities of like nature are of no dimensions, is that their dimensional representation according to the above conventions simplifies down to unity. No other physical quantities are, however, dimensionally represented by unity. We may, therefore, say that *tensors* &c. are also physical quantities, and therefore concrete, and that since every physical quantity consists of two factors, namely a pure number and a concrete, we may regard a pure number when occurring alone in a physical relation as a physical quantity whose concrete factor is a *tensor*. Since, however, the term "concrete" is reserved for those quantities which cannot be completely specified by pure numbers, we come ultimately to regard a tensor as a pure number, and the versor as a concrete quantity.

---

* See "The Multiplication and Division of Concrete Quantities," Prof. A. Lodge, 'Nature,' July 19, 1888.

If we substitute for α, β, γ in the above X, Y, Z respectively, we get the same expressions for lengths, areas, volumes, and angles as already obtained in the dimensional formulæ. Hence, we see that the conventions made as to the dimensional representation of the various quantities, namely, that areas, volumes, and angles should be represented by products and quotients of *different* vector lengths instead of by powers and quotients of a *given one*, are justified by the fact that they are consistent with the meanings of products and quotients of rectangular vectors. For the products and powers of parallel vectors can never represent areas or volumes, and their ratios can be nothing but pure numbers.

Dimensional formulæ may be conveniently expressed in this way, that is, by taking three rectangular vectors in space as axes of reference. We may, of course, suppose X, Y, Z to be the vectors. Then the formulæ already deduced will be unaltered, except that the proper signs have to be inserted according to the order the products and quotients are performed. The great advantage of thus supposing X, Y, Z to be vectors instead of mere Cartesian lengths is that the distinction between scalar and vector quantities is made apparent by the formula. Thus, X^2, Y^2, Z^2, or R^2 (R being any vector) are *scalars*; X, Y, Z, XY, YZ, ZX, XY^{-1}, YZ^{-1}, ZX^{-1} and their reciprocals are *vectors*; and (XYZ) a scalar. To make the formulæ exhibit at the same time the numerical dependence of the derived upon the fundamental units, we may put $X = iL$, $Y = jL$, $Z = kL$, i, j, k being quadrantal versors and L the *scalar unit length*. Thus, neglecting signs, we have:—

$$\text{Force} = MXT^{-2} = M(iL)T^{-2}.$$

$$\text{Work} = MX^2T^{-2} = M(iL)^2T^{-2} = ML^2T^{-2}.$$

$$\text{Couple} = MXYT^{-2} = MijL^2T^{-2} = MkL^2T^{-2}.$$

$$\text{Energy per unit volume,} = (ML^2T^{-2})L^{-3} = ML^{-1}T^{-2},$$

$$\text{for volume} = (ijkL^3) = L^3,\ ijk = 1.$$

$$\text{Pressure} = \frac{\text{MXT}^{-2}}{\text{YZ}} = \frac{\text{M}i\text{LT}^{-2}}{jk\text{L}^2} = \text{ML}^{-1}\text{T}^{-2}, \ (i=jk).$$

$$\text{Tangential force per unit area} = \text{MY}^{-1}\text{T}^{-2} = \text{M}(j\text{L})^{-1}\text{T}^{-2}.$$

$$\text{Angle} = \text{XY}^{-1} = (i\text{L})(j\text{L})^{-1} = \frac{i}{j} = \frac{ij}{j^2} = k.$$

$$\text{Work done by a couple} = \text{MXYT}^{-2}(\text{XY}^{-1}) = (\text{M}ij\text{L}^2\text{T}^{-2})k$$
$$= \text{ML}^2\text{T}^{-2}, \ (ijk=1).$$

Thus, the indices of M, L, and T indicate the *numerical* dependence of the derived upon the fundamental units, the formulæ being identical in this respect with the ordinary ones, while the directional properties of the quantities are clearly differentiated by i, j, k.

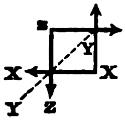
Multiplying the numerator and denominator of a formula by X, Y, or Z makes no change in its physical interpretation. For physical quantities differ from each other only in their numerical dependence upon the fundamental units, and their "space relations," the former being expressed by the scalar part of the formula L, M, T, the latter by the vector part i, j, k. But if X, Y, Z be equal rectangular vectors, the tensors of $\frac{\text{X}}{\text{X}}, \frac{\text{Y}}{\text{Y}}, \frac{\text{Z}}{\text{Z}}$ are unity, and their versors are zero. Hence, no change can be made in the scalar or vector part of any formula into which they may be multiplied. Thus:—

$$\text{Force} = \text{MXT}^{-2} = \frac{\text{MX}^2\text{T}^{-2}}{\text{X}}$$
$$= \text{Space rate of variation of energy along X.}$$

$$\text{Force} = \text{MXT}^{-2} = \frac{\text{MXZT}^{-2}}{\text{Z}} = \text{Couple per unit area.}$$

$$\text{Tangential force per unit area} = \text{MY}^{-1}\text{T}^{-2} = \frac{\text{MXT}^{-2}}{\text{XY}}$$
$$= \text{Force along X in plane XY} = \frac{\text{MZT}^{-2}}{\text{YZ}}$$
$$= \text{Force along Z in plane YZ.}$$

Fig. 4.



These are the two components of a shearing-stress referred to a unit cube. Multiplying by $\frac{\text{XZ}}{\text{XZ}}$ we get $\frac{\text{MXZT}^{-2}}{\text{XYZ}}$ = Couple per unit volume.

Now, a shearing-stress must be of the nature of a couple, for a shear is of the nature of an angle, and the product of the stress into the shear is work done per unit volume.

$$\text{Surface tension} = \frac{MXT^{-2}}{Y} = \frac{MX^2T^{-2}}{XY} = \text{Energy per unit area.}$$

$$\text{Pressure} = \frac{MXT^{-2}}{YZ} = \frac{MX^2T^{-2}}{XYZ} = \text{Energy per unit volume.}$$

Since, as above shown, we obtain the same dimensional formulæ by Cartesian and vectorial methods, we may attach to X, Y, Z and their products and quotients in these formulæ either purely vectorial or purely Cartesian meanings just as we please: in both cases, the formulæ represent the same physical identities. When X, Y, Z are vectors the formulæ express the directional properties of the corresponding quantities. [An inspection of a formula shows this, for the products and quotients of two rectangular vectors are vectors directed normally to the planes containing the original vectors, and the products and quotients of parallel vectors are scalar]. Thus: Pressure $MX(YZ)^{-1}T^{-2}$, scalar, for YZ is directed parallel to X, hence $\frac{X}{YZ}$ is scalar. Couple $MXZT^{-2}$, a vector directed along Y, &c.

In deducing the dimensions of electromagnetic quantities it will be necessary to start with the dimensions of energy, or energy per unit volume. Using Cartesian units of length this is MX^2T^{-2}, MY^2T^{-2}, MZ^2T^{-2}, or MR^2T^{-2}, as the case may be; and since we shall have to deal with connected equations between the various quantities, the particular form to be used becomes of importance. Of course, energy being a scalar quantity, the above do not differ *essentially*: the difference arises only from the different ways (the different dynamical reactions) by which the expressions are derived. It might be more convenient to put $X=iL$, $Y=jL$, $Z=kL$, then the dimensions of the energy of the medium become ML^2T^{-2}. There is thus nothing to indicate how or with reference to what dynamical reactions the expression is derived. In what follows, however, Cartesian expressions will still be used though more cumbrous and involving more explanation. They may be immediately converted into the above. The energy of the medium will be expressed in terms of an *instantaneous* linear displacement R upon which our conceptions of the electromagnetic displacements at a point, whatever their nature may be, must ultimately depend. By keeping R, at first, separate from X, Y, and Z, the formulæ gain in generality, and by retaining R, X, Y, Z instead of their equivalent vector forms rL, iL, jL, kL, the dynamical connexion between the various quantities is more clearly expressed.

The quantity π enters prominently into electromagnetic relations, and it becomes necessary at the outset to determine in what manner it is to be dealt with. This subject has been discussed by Mr. Oliver Heaviside (Elec. October 16th and 30th, 1891), and his conclusions may be briefly summarized as follows :—

If m and q be point sources of induction and displacement respectively, the measure of the induction and displacement at a distance r from the source (if the fluxes emanate isotropically) is

$$\mathbf{B}=\left(\frac{m}{4\pi r^2}\right), \quad \mathbf{D}=\left(\frac{q}{4\pi r^2}\right),$$

where **B** and **D** are the densities of the fluxes over spherical surfaces enclosing the sources. And, similarly, the density of any radial flux at a point should be estimated by the total flux through a spherical surface having its centre at the source and passing through the given point divided by the surface. Writing $\mathbf{B}=\mu\mathbf{H}$, and $\mathbf{D}=k\mathbf{E}$, where μ and k express physical properties of the medium, we have

$$\mathbf{H}=\frac{1}{\mu}\left(\frac{m}{4\pi r^2}\right), \quad \mathbf{E}=\frac{1}{k}\left(\frac{q}{4\pi r^2}\right).$$

Now, **H** and **E** express the strengths of the fields produced by the fluxes m and q at distances r from the source. Hence $\frac{m}{\mu}$ and $\frac{q}{k}$ express the strengths of the sources. Again, multiplying the above by m and q respectively we get

$$m\mathbf{H}=\frac{1}{\mu}\left(\frac{m^2}{4\pi r^2}\right), \quad q\mathbf{E}=\frac{1}{k}\left(\frac{q^2}{4\pi r^2}\right).$$

But $m\mathbf{H}$ is the force experienced by a pole m when placed in a field of strength $\mathbf{H}$, and similarly for $k\mathbf{E}$. Hence

$$F_m = m\mathbf{H} = \frac{1}{\mu}\left(\frac{m^2}{4\pi r^2}\right), \quad F_q = q\mathbf{E} = \frac{1}{k}\left(\frac{q^2}{4\pi r^2}\right),$$

where F_m and F_q are respectively the forces between two poles m, or two charges q, at distances r apart. In other words, since in expressing the force between two poles or two charges we have to regard each pole or charge as an *isolated point source of displacement*, we should regard the one pole or charge as producing radially a field of given strength, and then express the force experienced by the other when placed in this field.

If, now, the unit pole and the unit charge be defined respectively by the relations

$$m = r\sqrt{4\pi\mu F_m}, \qquad q = r\sqrt{4\pi k F_q},$$

instead of, as usual,

$$m = r\sqrt{\mu F_m}, \qquad q = r\sqrt{k F_q},$$

the effect is to *redistribute* π in electromagnetic equations as a whole. It is found, however, that all those relations into which it is now made to enter depend upon and involve the consideration of circuital or radial fluxes, and π obviously enters as a plane or solid angle in connexion with circles and spheres of reference. It has thus a definite physical meaning, and is always definitely related to the other quantities in the relation. On the other hand, in the case of the relations from which it is removed, it previously entered only because of its suppression elsewhere, and had no fundamental connexion with the other quantities.

The relation between the new and the old units thus defined are given in the papers above referred to. For the purpose of the present paper it is sufficient to notice that the relation

$$4\pi u \partial y \partial z = \left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right)\partial y\, \partial z$$

becomes

$$u \partial y \partial z = \left(\frac{\partial\gamma}{\partial y} - \frac{\partial\beta}{\partial z}\right)\partial y \partial z:$$

$p = 4\pi m$ becomes $p = m$: $\mathbf{B} = \mu_0\mathbf{H} + 4\pi\mathbf{I}$ becomes $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{I}$, so that **B** and I are identical; the electrostatic energy of the medium $\left(\frac{\mathbf{ED}}{4\pi}\right)$ becomes **ED**, thus harmonizing with **BH**.

Since, in these relations, π essentially preserves its primitive geometrical or trigonometrical meaning, its insertion or suppression implies the insertion or suppression of a *concrete quantity*, and not merely a number, so that the natural relations between the various quantities are affected. For this reason, therefore, the "rational units," given by Mr. Oliver Heaviside, in which π is primarily associated with the radial and circuital fluxes of the field, will alone be used in what follows.

The relations made use of in deducing the dimensions of electromagnetic quantities are of three kinds.

I. Relations between quantities of the same kind, either purely electrical or purely magnetic.—These are :—

(*a*) Electrical.

1. $\mathbf{D}=k\mathbf{E}$.
2. $\mathrm{C}=\frac{\partial}{\partial t}.e$.
3. $e=$Surface-integral of **D**.
4. $\mathbf{C}=\frac{\partial}{\partial t}.\mathbf{D}$.
5. $\mathrm{E}=$Line-integral of **E**.

Where **D**=electric displacement; k=specific inductive capacity; **E**=electrical force; C=electric current; e=quantity of electricity; **C**=current density; E=electromotive force or *voltage* of a closed circuit.

(*b*) Magnetic.

1. $\mathbf{B}\ =\mu\mathbf{H}$.
2. $\mathrm{C}_m=\frac{\partial}{\partial t}.m$.
3. $m\ =$Surface-integral of **B**.
4. $\mathbf{C}_m=\frac{\partial}{\partial t}.\mathbf{B}$.
5. $\mathrm{E}_m=$Line-integral of **H**.

B=magnetic induction; μ=specific magnetic capacity; **H**=magnetic force; m=quantity of magnetism. Mr Oliver

Heaviside designates E_m the *Gaussage* of a closed magnetic circuit to correspond with E, the *Voltage* of the corresponding electrical one, C_m and $\mathbf{C}_m$ being the magnetic currents.

II. Relations between quantities of different kinds.—These relations are embodied in the two laws of circuitation:

$$(a)\qquad \textit{Circuitation}\ \mathbf{H}=C=E_m=\frac{\partial}{\partial t}.e\,;$$

or, the *Gaussage* of a closed magnetic circuit measures the total electrical current through it.

$$(b)\qquad \textit{Circuitation}\ \mathbf{E}=C_m=E=\frac{\partial}{\partial t}.\,m\,;$$

or, the *Voltage* of a closed electrical circuit measures the total magnetic current through it.

The corresponding relations as given by Maxwell are

$$(a)\qquad u\partial y\,\partial z=\left(\frac{\partial\gamma}{\partial y}-\frac{\partial\beta}{\partial z}\right)\partial y\,\partial z,$$

u being the component current-density along x, and β, γ the components of magnetic force along y and z respectively, the whole being referred to an elementary magnetic circuit $(\partial y\,\partial z)$. [The 4π in the expression $4\pi u$ is dropped, as previously explained.]

$$(b)\qquad P=\left(c\frac{\partial y}{\partial t}-b\frac{\partial z}{\partial t}\right),$$

neglecting **A** and ψ. P is the component electrical force along x, and b and c the components of magnetic induction along y and z respectively.

III. Dynamical Relations.—These are relations between quantities whose product express the energy, or energy per unit volume of the medium.

eE : $m\Omega$: pC . . . Energy.
DE : **BH** : **AC** . . . Energy per unit volume.

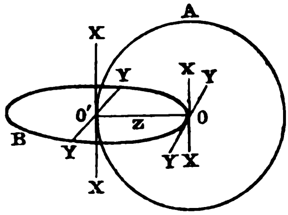
But $\mathbf{D}=k\mathbf{E}$, and $\mathbf{B}=\mu\mathbf{H}$. Hence

$k\mathbf{E}^2$: $\mu\mathbf{H}^2$ . . Energy per unit volume.

By means of these relations we can express in terms of M, X, Y, Z, T, and *one selected quantity* the dimensions of all the rest. The only useful cases, however, are those in which the selected quantities are either μ or k, for these express physical properties of the medium at a point, and are independent of the electromagnetic reactions going on there. The dimensions in terms of μ are obtained by starting with the relation $\mu \mathrm{H}^2=$Energy per unit volume; and similarly for the dimensions in terms of k.

Since the above dimensions have to be deduced by means of a connected system of equations, it becomes necessary to make a suitable choice of axes of reference. Let X be the axis of the electric displacement, and Y that of the magnetic displacement at a point in the medium. For an isotropic medium (the only case we have at present to consider) these are mutually at right angles, and Z is at right angles to both, being the intersection of the electric and magnetic equipotential surfaces. Let this relation between the directions of the axes of reference and the displacements hold for every point of the medium, so that the axes constitute an instantaneous system at every point. In passing therefore from point to point in the medium, and for different epochs at the same point, the axes and the displacements preserve their *relative* directions, while their *absolute* direction in space in general alters.

Fig. 5.



Let AO′ be a closed electric circuit, and BO a corresponding closed magnetic circuit, both being circles in planes at right angles to each other. Taking instantaneous axes as above, every element of the circuit AO′ is ∂x, and of the circuit BO is ∂y, while an element of the intersection of the planes of the circuits is ∂z. The length of the circuit AO′ is $\Sigma\partial x$, and of the circuit BO, $\Sigma\partial y$, while the surfaces of the circuits are ultimately $\Sigma(\partial x\partial z)$, and $\Sigma(\partial y\partial z)$. We have therefore :—

1. Circuitation $\mathbf{H}=\Sigma(\mathbf{H}\partial y)=\mathrm{C}$.
2. Circuitation $\mathbf{E}=\Sigma(\mathbf{E}\partial x)=\mathrm{E}$.

3. Surface-integral of $=\mathbf{D}\Sigma(\mathbf{D}\partial y\partial z)=e$.
4. Surface-integral of $\mathbf{B}=\Sigma(\mathbf{B}\partial x\partial z)=m$.

To express these dimensionally, we have to neglect the summation Σ, and substitute for ∂x, ∂y, ∂z respectively X, Y, Z. The relations then become :—

1. $\mathbf{D}=k\mathbf{E}$.
2. $C=eT^{-1}$.
3. $e=\mathbf{D}(YZ)$.
4. $\mathbf{C}=\mathbf{D}T^{-1}$.
5. $E=\mathbf{E}(X)=C_m=mT^{-1}=\mathbf{B}(XZ)T^{-1}$.
6. $\mathbf{B}=\mu\mathbf{H}$.
7. $C_m=mT^{-1}=E$.
8. $m=\mathbf{B}(XZ)$.
9. $\mathbf{C}_m=\mathbf{B}T^{-1}$.
10. $E_m=\mathbf{H}(Y)=C=eT_{-1}=\mathbf{D}(YZ)T^{-1}$.

The energy of the medium at any point may be expressed by

$$\Sigma m(\dot{x}^2+\dot{y}^2+\dot{z}^2)=\Sigma(m\dot{r}^2),$$

where r is the instantaneous linear displacement upon which both the electric and magnetic displacements at that point depend, for the two laws of circuitation express that the electric and magnetic displacements at a point in the medium are not independent, but originate from the same dynamical cause. Expressed dimensionally, this becomes

$$M(X^2+Y^2+Z^2)T^{-2} \text{ or } MR^2T^{-2},$$

and the dimensions of energy per unit volume are

$$MR^2(XYZ)^{-1}T^{-2}.$$

Since the axes of reference (X, Y, Z) and the displacement R are both instantaneous with respect to any point, the direction of R must be definitely related to the axes, the relation being dependent upon the dynamical mechanism of the field. It will be convenient, however, to express the dimensions of electromagnetic quantities, first, in terms of M, X, Y, Z, T and R; and afterwards to determine the relation between the direction of R and the axes. The fact that the formulæ of some of the quantities involve R, while others do not, then simply means that in the former cases the corresponding quantities depend upon and involve the instantaneous linear displacement specified by R ; while in the other cases they are independent of the displacement and express only physical properties of the medium.

I. Electromagnetic System :—

$$\mu\mathbf{H}^2=MR^2T^{-2}(XYZ)^{-1}.$$

Hence

1. $\mathbf{H}=\mu^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(XYZ)^{-\frac{1}{2}}]$.
2. $\mathbf{B}=\mu\mathbf{H}=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(XYZ)^{-\frac{1}{2}}]$.
3. $m=p=\mathbf{B}(XZ)=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]$.
4. $E=mT^{-1}=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-2}(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]$.
5. $\mathbf{E}=E(X^{-1})=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-2}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]$.
6. $\mathbf{A}=\mathbf{E}T=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]$.
7. $C=\mathbf{H}Y=\mu^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{\frac{1}{2}}Z^{-\frac{1}{2}})]$.
8. $\mathbf{C}=C(YZ)^{-1}=\mu^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.
9. $e=CT=\mu^{-\frac{1}{2}}[M^{\frac{1}{2}}R(X^{-\frac{1}{2}}Y^{\frac{1}{2}}Z^{-\frac{1}{2}})]$.
10. $\mathbf{D}=\mathbf{C}T=e(YZ)^{-1}=\mu^{-\frac{1}{2}}[M^{\frac{1}{2}}R(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.

From 5 and 10 we have

$$k=\frac{\mathbf{D}}{\mathbf{E}}=\mu^{-1}[Z^{-2}T^2],$$

and substituting $k^{-1}[Z^{-2}T^2]$ for μ all through in the above we get for each quantity its electrostatic dimensions, the results being identical with those obtained directly as below.

II. Electrostatic System :—

$$k\mathbf{E}^2=MR^2T^{-2}(XYZ)^{-1}.$$

1. $\mathbf{E}=k^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(XYZ)^{-\frac{1}{2}}]$.
2. $\mathbf{D}=k\mathbf{E}=k^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(XYZ)^{-\frac{1}{2}}]$.
3. $e=\mathbf{D}(YZ)=k^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{\frac{1}{2}}Z^{\frac{1}{2}})]$.
4. $C=eT^{-1}=k^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-2}(X^{-\frac{1}{2}}Y^{\frac{1}{2}}Z^{\frac{1}{2}})]$.
5. $\mathbf{C}=C(YZ)^{-1}=k^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-2}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.

6. $\mathbf{H}=C(Y^{-1})=k^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-2}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]$.
7. $E=\mathbf{E}(X)=k^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.
8. $m=p=ET=k^{-\frac{1}{2}}[M^{\frac{1}{2}}R(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.
9. $\mathbf{B}=m(XZ)^{-1}=k^{-\frac{1}{2}}[M^{\frac{1}{2}}R(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{3}{2}})]$.
10. $\mathbf{A}=\mathbf{E}T=k^{-\frac{1}{2}}[M^{\frac{1}{2}}R(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$.

From 6 and 9 we have

$$\mu=\frac{\mathbf{B}}{\mathbf{H}}=k^{-1}[Z^{-2}T^{2}]\ ;$$

and by substituting $\mu^{-1}[Z^{-2}T^{2}]$ all through for k in the above, we recover the electromagnetic dimensions previously deduced.

The following are examples of the manner these formulæ work out :—

1. The velocity of propagation of an electromagnetic disturbance in the medium is given by

$$V=\frac{1}{\sqrt{k\mu}}=ZT^{-1}\ \text{(dimensionally)},$$

which is of the dimensions of a velocity along Z, the normal to the plane of displacements (XY).

2. The flux of energy at a point is given by

$$W=(\mathbf{E}\mathbf{H})\ ;$$

and expressing $\mathbf{E}$ and $\mathbf{H}$ dimensionally in terms of μ or k we get

$$\frac{1}{T}\left(\frac{MR^{2}T^{-2}}{XY}\right)=\left(\frac{MR^{2}T^{-2}}{XYZ}\right)\frac{Z}{T}=[W],$$

the bracketed factor being of the dimensions of energy per unit volume, and the other a velocity in the direction Z, the axis of flux of energy.

3. The force between two poles m is given by

$$F_m=(m\mathbf{H})=\frac{1}{\mu}\left(\frac{m^2}{4\pi r^2}\right),$$

where $\mathbf{H}$ is the strength of the field produced at the one by the other. Expressed dimensionally, this becomes

$$F_m=\left(\frac{MR^{2}T^{-2}}{Y}\right),$$

which is of the dimensions of the space rate of variation of energy along Y, that is, the *force* along Y, as is evident from the equivalent algebraic expression

$$m\frac{\partial^2 y}{\partial t^2}=\frac{\partial}{\partial y}\Sigma(m\dot{r}^2).$$

Putting $R=rL$, and $Y=jL$, where r and j are unit vectors in the directions R and Y, and L the *scalar* unit length, we have

$$[F_m]=\frac{MR^2T^{-2}}{Y}=\frac{M(rL)^2T^{-2}}{jL}=-\frac{ML^2T^{-2}}{jL}=-\frac{MLT^{-2}}{j}$$
$$=M(jL)T^{-2}=MYT^{-2}.$$

Thus, the force between two poles is in the direction of magnetization. The reason why the force is expressed in terms of the energy of the system is that it is a *mechanical* force arising in some way from the mutual reaction between matter and the medium. The quantities m and **H** in terms of which the force between the two poles is expressed refer to the medium alone, and since nothing is known as to the relation between the medium and matter, the relation above expresses only the *resultant* reaction taking place, namely that there is a space rate of variation of the energy of the system along Y, the direction of the force. The same remarks apply to the cases below.

4. The force between two charges q is given by

$$F_q=(q\mathbf{E})=\frac{1}{k}\left(\frac{q^2}{4\pi r^2}\right),$$

or dimensionally $[F_q]=\dfrac{MR^2T^{-2}}{X}$, the space rate of variation of the energy of the system they constitute along X, the direction of the force.

5. The force per unit volume between two elements of conductors carrying currents is given by $F_c=\mathbf{BC}$, where **C** is the current-density in the one, and **B** the induction at this one due to the other. Expressed dimensionally, this becomes

$$[F_c]=\frac{MR^2T^{-2}}{XYZ^2}=\left(\frac{MRT^{-2}}{XYZ}\right)\frac{1}{Z},$$

which is of the dimensions of space rate of variation of energy per unit volume along Z, the direction of the force, for the conductors move at right angles to their lines of force which are circles about them.

Whatever be the dimensions of μ those of k must be given by $\mu^{-1}[Z^{-2}T^{2}]$. Now, the ratio of the dimensions of the same quantity expressed in the two systems is a power of $\mu k[Z^{2}T^{-2}]$. Hence, all possible dimensional values of μ together with the corresponding ones for k will, when substituted in the formulæ, bring the two systems into accord. Of these, however, there must be one pair, and only one pair, which give rise to dimensions whose interpretations are physically real.

It is, of course, impossible to determine from purely dimensional considerations what this particular pair must be, for this would imply a knowledge of the dynamical nature of electromagnetism. It *is* possible, however, to assign to μ and k dimensions fulfilling certain *assumed* conditions. The dimensions of the remaining quantities then become unique, and we may, by deducing the physical interpretations of the formulæ, pass on to the quantities they then represent. In this way, the formulæ in terms of μ and k may be utilized as means for tracing out in detail the various analogies between electromagnetism and dynamics. For every dimensional value of μ and k we thus obtain a perfectly connected dynamical analogue of electromagnetism, which may or may not be rational and intelligible as a whole, and whose relation to physical reality depends upon the imposed conditions.

For the purpose of the present paper, let us impose upon the dimensions of μ and k the condition that the indices of the fundamental units in their formulæ are not to be higher or lower than those found in the formulæ of ordinary dynamical units. This is, of course, a *purely arbitrary condition*, and simply expresses that the dynamical analogues to be traced out are restricted to those which are of a simple, natural, and intelligible character. All other cases are purposely excluded, not, of course, as a matter of necessity, but as a matter of convenience, for nothing would be gained by the introduction of formulæ whose interpretation would be obscure and unintelligible in our present state of knowledge. Let us therefore proceed to deduce from dimensional considerations the various analogies between electromagnetism and dynamics, *subject to the arbitrary condition imposed above.*

In the case of ordinary dynamical units we notice that

1. No fractional powers of the fundamental units occur.
2. The indices of M are never higher than ± 1.
3. „ „ X, Y, Z „ „ „ ± 2.
4. „ „ T „ „ „ ± 3.

This, therefore, is the range within which μ must be found subject to the above conditions.

[Of course, if the fundamental units be not L, M, T, but some of the *now* derived units, then fractional powers may occur. Thus, if V (volume) be a fundamental unit, the dimensions of length are $V^{\frac{1}{3}}$, and of area $V^{\frac{2}{3}}$, &c. So long, however, as L, M, T, are fundamental units, we cannot expect fractional powers to occur. For *length*, or space of one dimension, is the simplest conception of space which we can form, while *time* and *mass* (not necessarily that of matter, but *tangibleness* in general) are fundamental conceptions beyond which we cannot go. Now, all dynamical conceptions are built up ultimately in terms of these three ideas, mass, length, and time, and since the process is synthetical, building up the complex from the simple, it becomes expressed in conformity with the conventions of Algebra by *integral powers* of L, M, T. The analytical process, that is, splitting up a complex conception into its ultimate constituents (evolution in Algebra), becomes expressed according to the same conventions by fractional powers, *e. g.* $L=(V)^{\frac{1}{3}}$, $[\text{Area}]=(V)^{\frac{2}{3}}$, &c. But, obviously, if mass, length, and time are to be *ultimate* physical conceptions, we cannot give interpretations to fractional powers of L, M, and T, because we cannot analyse the corresponding ideas to anything simpler. We should thus be unable, according to any physical theory, to give interpretations to formulæ involving fractional powers of the fundamental units. Our only hope in such cases would be that the units themselves might cease to be fundamental. On the other hand, the building up of a derived from fundamental units is always a simple process, nothing but *whole units* of the latter kind being involved. We should thus *expect*, both from the algebraical meanings of integral and fractional powers, and from the manner physical units are derived, to find the dimensional formulæ of physical quantities free from fractional powers. That the formulæ of all dynamical quantities, that is all the *absolute* formulæ we are acquainted with, are of this kind, is an argument in favour of this view.

A vector length has two properties, direction and magnitude. When squared, a vector length becomes scalar. A vector enters into physical relations either as a vector (having direction and magnitude) or as a scalar (having magnitude only). Hence we may say that the first and second powers of a vector, X and X^2 (say), represent the two different ways in which it enters all physical relations. For all other integral powers differ from these only in scalar magnitude. If the dimensional formulæ are to express different *natural relations* between quantities, then, so far as the unit length is concerned (the representative of space), all these relations are involved in X and X^2, X being a vector. If this be true, then the fact that the indices of vector lengths, in the formulæ of dynamical quantities, are never higher than ± 2 becomes explained. In a similar manner, we may explain the fact that the indices of M are always ± 1, for here no new idea is introduced by M^2 as in the case of vectors: it differs from M only in scalar magnitude, and still conveys the same physical idea, namely *inertia*. It is widely different in the case of time, for every new *negative* power of T introduces a new physical idea with respect to time—that is, it introduces the conception of a new *time-flux*.

It is not necessary, however, to postulate anything as to these matters. It is sufficient for our present purpose to know what the limits of the indices of the fundamental units are in the case of known dynamical quantities, without having to account for the same. The above are only suggestions.]

Let the dimensions of μ be

$$[\mu] = M^m\, R^r\, T^t\, (X^x, Y^y, Z^z),$$

then those of k are

$$[k] = M^{-m}\, R^{-r}\, T^{-t+2} (X^{-x}\, Y^{-y}\, Z^{-z-2}).$$

Since the indices of M, X, Y, Z in the dimensional formulæ are odd multiples of $\frac{1}{2}$, their indices in the dimensions of μ must be *odd*, otherwise, on substituting for μ and k in these formulæ their dimensional values, the formulæ would not be rationalized. For a similar reason, the indices of R and T must be *even*. Hence, the indices of M, X, Y, Z must be ± 1, and of R, 0, or ± 2. In the case of Z, $+1$ is not admissible, otherwise we should have Z^{-3} in the case of k. Thus, Z must enter both μ and k as Z^{-1}. In the case of T the possible values are 0, ± 2, ± 4, ... $\pm 2n$. These values may be tabulated thus :—

[μ]		[k]	
(a)	(b)	(a')	(b')
0	...	$+2$	...
$+2$	-2	0	$+4$
$+4$	-4	-2	$+6$
$+6$	-6	-4	$+8$
. . . .	. . . .	. . . .	. . . .
$+2n$	$-2n$	$-2(n-1)$	$+2(n+1)$

Under a are given the *positive* possible values for t in μ, under a' the corresponding values in the case of k. Under b, the *negative* values in the case of μ, and under b' those corresponding in the case of k. The only cases necessary to be considered are T^0 and T^2, for, as seen from the table, all other possible values of T lead to dimensional formulæ involving powers of T higher than ± 4.

The dimensions of μ and k involve only X, Y, Z, M and T. This is of course obvious if R has to coincide ultimately with X, Y, or Z. If R is not ultimately to coincide with X, Y, or Z, it cannot enter into the formulæ of μ and k. For R can enter into the formula of μ only as R^2 or R^{-2}. In the former case, $\mu^{\frac{1}{2}}$ contains R, and $\mu^{-\frac{1}{2}}$ contains R^{-1}. But $\mu^{\frac{1}{2}}$ is a factor of the formulæ of m, $\mathbf{B}$, $\mathbf{E}$, E, and $\mu^{-\frac{1}{2}}$ of e, $\mathbf{D}$, C, $\mathbf{C}$, $\mathbf{H}$; and since R is a factor of the formulæ of all the above quantities, we should have :—

i. m, $\mathbf{B}$, $\mathbf{E}$, E, containing R in their dimensional formulæ,
ii. e, $\mathbf{D}$, $\mathbf{H}$, C, $\mathbf{C}$, containing R^0 :

thus indicating that some of the forces and fluxes of the field would be dependent upon R, the others independent of it. But this is obviously impossible, since R is *the* instantaneous linear displacement at a point in the field upon which depend all the electromagnetic forces and fluxes at that point. In a similar way, if μ involves R^{-2}, *m*, **B**, **E**, E would be independent of R, and *e*, **D**, **H**, C, **C** dependent upon it, leading to the same conclusion as before. Thus, μ and *k* must be quantities independent of the electromagnetic reactions (specified by R) which may be going on in the field, as is otherwise evident since μ and *k* express physical properties of the medium. We have therefore to consider the different ways the dimensions of μ (say) can be built up from M, X, Y, Z, and T, subject to the conditions already laid down.

The possible dimensional values for μ are to be obtained from $M^{\pm 1} X^{\pm 1} Y^{\pm 1} Z^{-1} T^{0, +2}$, by forming all the possible combinations of the quantities taken all together. The possible cases are :—

I. 1. $MX^{-1}Y^{-1}Z^{-1}$.
2. $MXYZ^{-1}$.
3. $MYY^{-1}Z^{-1}$.
4. $MX^{-1}YZ^{-1}$.

II. 1. $MX^{-1}Y^{-1}Z^{-1}T^2$.
2. $MXYZ^{-1}T^2$.
3. $MXY^{-1}Z^{-1}T^2$.
4. $MX^{-1}YZ^{-1}T^2$.

III. 1. $M^{-1}X^{-1}Y^{-1}Z^{-1}$.
2. $M^{-1}XYZ^{-1}$.
3. $M^{-1}XY^{-1}Z^{-1}$.
4. $M^{-1}X^{-1}YZ^{-1}$.

IV. 1. $M^{-1}X^{-1}Y^{-1}Z^{-1}T^2$.
2. $M^{-1}XYZ^{-1}T^2$.
3. $M^{-1}XY^{-1}Z^{-1}T^2$.
4. $M^{-1}X^{-1}YZ^{-1}T^2$.

There are thus sixteen cases, and since for each value of μ we have up to the present supposed that R may ultimately coincide with X, Y, or Z, or with neither, the different systems which may be deduced from the above are sixty-four in number. There are other conditions, however, which enable us to reduce this number.

The quantities *m*, E, C, and *e* are scalar, and **B**, **E**, **C**, **D** vectors. Hence, the dimensional values of μ must be such that when substituted in the dimensional formulæ, the scalar quantities remain scalars, and the vector quantities vectors properly directed with respect to the dimensional axes. Since all quantities are scalar so far as M and T are concerned, it is only necessary to examine how the scalar and vector character of the quantities depend upon X, Y, Z. To do this, we may take the relation $[m]=\mu^{\frac{1}{2}}[\mathrm{MRT}^{-1}(\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}})]$, and substitute for μ successively the four factors $(\mathrm{XYZ})^{-1}$, (XYZ^{-1}), $(\mathrm{XY}^{-1}\mathrm{Z}^{-1})$, $(\mathrm{X}^{-1}\mathrm{YZ}^{-1})$. The conclusions deduced for m must obviously hold for the other quantities. Thus :—

1. Let $\mu^{\frac{1}{2}}$ contain $\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}}$. Then $[m]$ contains

$$(\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}})\ \mathrm{R}\ (\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}})=\mathrm{RY}^{-1}.$$

This becomes scalar only when R coincides with Y.

2. Let $\mu^{\frac{1}{2}}$ contain $\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}}$. Then $[m]$ contains

$$(\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}})\ \mathrm{R}\ (\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}})=\mathrm{RX}.$$

This becomes scalar only when R coincides with X.

3. Let $\mu^{\frac{1}{2}}$ contain $\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}}$. Then $[m]$ contains

$$(\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}})\ \mathrm{R}\ (\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}})=\mathrm{R}\times\mathrm{Y}^{-1}.$$

This becomes scalar when R coincides with Z.

4. Let $\mu^{\frac{1}{2}}$ contain $\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}}$. Then $[m]$ contains

$$(\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}})\ \mathrm{R}\ (\mathrm{X}^{\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}})=\mathrm{R}.$$

This cannot become scalar.

Thus, in order to render the scalar quantities scalar, R must ultimately coincide with X, Y, or Z. There are thus three cases to be considered :—

1. When R coincides with Y, μ contains $(\mathrm{XYZ})^{-1}$.
2. „ „ „ X, „ (XYZ^{-1}).
3. „ „ „ Z, „ $(\mathrm{XY}^{-1}\mathrm{Z}^{-1})$.

But, according to the electromagnetic theory of light, R cannot coincide with Z, for XY is the wave-front of an instantaneous plane-polarized disturbance at a point in the medium—the disturbance ultimately originating in the displacement R, and since the medium is isotropic, the linear displacement specified by R can have no components along Z. We are thus restricted to the two cases where R coincides with Y, and the dimensions of μ involve $(\mathrm{XYZ})^{-1}$, or where R coincides with X, and the dimensions of μ involve XYZ^{-1}. In the former case, the instantaneous linear displacement R of the medium at any point is in the plane of polarization (the plane of magnetic displacement), magnetic energy is kinetic, and electrical energy potential. In the latter case, the displacement R is at right angles to the plane of polarization, magnetic energy is potential, and electrical energy kinetic. Physicists are not yet agreed as to the interpretation of Weiner's results, so that no arguments as to the direction of the instantaneous displacement of the medium at a point can be based upon his experiments. We have, therefore, to suppose that R may lie along either Y or X, and the possible systems become reduced to eight, as below:—

1. $M(XYZ)^{-1}$. 2. $M^{-1}XYZ^{-1}T^2$.

3. $M(XYZ^{-1})$. 4. $M^{-1}X^{-1}Y^{-1}Z^{-1}T^2$.

5. $M(XYZ)^{-1}T^2$. 6. $M^{-1}XYZ^{-1}$.

7. $M(XYZ^{-1})T^2$. 8. $M^{-1}X^{-1}Y^{-1}Z^{-1}$.

If the dimensions of μ be (1) those of k are (2), and if those of k are (1), the dimensions of μ are (2), and similarly for the other three pairs.

The dimensional formulæ 5 and 7 are unintelligible, for we have no dynamical units involving positive powers of both M *and* T, and it is difficult to give to such formulæ an unintelligible interpretation. The same difficulty appears in 6 and 8, for if one of these be the dimensional formula for μ, the corresponding one for k is 5 or 7. Apart, however, from this difficulty, these formulæ 5, 6, 7, and 8 lead to fluid theories of electromagnetism. Thus, from 5, we get for the dimensions of m:—

$$[m]=\mu^{\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})]=M^{\frac{1}{2}}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})R(X^{\frac{1}{2}}Y^{-\frac{1}{2}}Z^{\frac{1}{2}})$$
$$M^{\frac{1}{2}}T^{-\frac{1}{2}}T^{\frac{1}{2}}=MRY^{-1}=M \text{ (since R here must coincide with Y).}$$

Similarly, from 7 we have $[m]=MX^2$. Again, from 6 we get $[e]=M$, and from 8, $[e]=MY^2$. Thus, we have to suppose *m* to be a quantity of the nature of *mass* or *moment* of *inertia*, and similarly for *e*. In both cases, *m* and *e* would be *properties* of the medium depending upon its inertia, instead of being parts of the electromagnetic reactions going on in the medium,—an electrical current would thus be a quantity of the nature of momentum, which is inconsistent with Maxwell's theory of electromagnetism. Other and not less serious difficulties will be met with if an attempt is made to develop their interpretations more fully.

Similar remarks apply to the systems arising from 3 and 4. Thus, from 3 we have

$$[e]=\mu^{-\frac{1}{2}}[\mathrm{M}^{\frac{1}{2}}\mathrm{R}(\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}})]$$
$$=\mathrm{M}^{-\frac{1}{2}}\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{-\frac{1}{2}}\mathrm{Z}^{\frac{1}{2}}\mathrm{M}^{\frac{1}{2}}\mathrm{R}\mathrm{X}^{-\frac{1}{2}}\mathrm{Y}^{\frac{1}{2}}\mathrm{Z}^{-\frac{1}{2}}=\mathrm{R}\mathrm{X}^{-1}=1,$$

since R here coincides with X. Thus, e is of the dimensions of a *volume-strain*, the ratio of two identical concretes. Similarly, we get from 4, $[m]=1$, also a volume-strain. Again, in the former case we have $\mathbf{D}=e(\mathrm{YZ})^{-1}$ $=$volume-strain per unit area. The two currents C and **C** become volume-strain per unit time (T^{-1}, or $\mathrm{Y}^{-1}\mathrm{Z}^{-1}\mathrm{T}^{-1}$). Electrical force **E** becomes MXT^{-2}, and voltage E $=\mathrm{MX}^2\mathrm{T}^{-2}$ (energy). Since $m=\mathrm{ET}$, m becomes $\mathrm{MX}^2\mathrm{T}^{-1}$, and magnetic moment $ml = \mathrm{MX}^2\mathrm{YT}^{-1}$. Again, **B** magnetic induction becomes $m(\mathrm{XZ})^{-1} = \mathrm{MXZ}^{-1}\mathrm{T}^{-1}$, and **H** magnetic force ($\mathrm{Y}^{-1}\mathrm{T}^{-1}$). Again, in the second case, we have $\mathbf{B}=m(\mathrm{XZ})^{-1}=$volume-strain per unit area; $\mathrm{E}=\mathrm{MT}^{-1}$ $=\mathrm{T}^{-1}=$volume-strain per unit time; $e=\mathrm{MY}^2\mathrm{T}^{-1}$; $\mathrm{C}=$ $\mathrm{MY}^2\mathrm{T}^{-2}$ (energy) and **C** (current density)$=\mathrm{MY}^2(\mathrm{YZ})^{-1}\mathrm{T}^{-2}$. These formulæ, although it may be possible to give them interpretations, are at present unintelligible, and do not suggest any connected relation between the various quantities.

The cause of the unintelligible character of these latter formulæ lies in the fact that the formula MXYZ^{-1} is not symmetrical with respect to the dimensional axes X, Y, Z. Since the above formula is independent of T, the corresponding quantity must be of the dimensions of some property of the medium depending ultimately upon the *inertia* of the medium alone. It is difficult, however, to conceive what property depending upon the inertia of the medium can involve the *three* dimensional axes unsymmetrically except the rotational inertia, or moment of inertia of a unit volume. The above formula, however, does not admit of such an interpretation. Thus, whether MXYZ^{-1} be the dimensional formula of μ or k, the results are either incapable of interpretation (as, for example, μ and k), or the interpretations are unintelligible.

Thus there are left only cases 1 and 2. If the dimensions of μ be those of 1, that is $M(XYZ)^{-1}$, μ becomes the inertia of unit volume, or the density of the medium, and magnetic energy is kinetic. If the dimensions of μ be 2, those of k are 1, and k becomes the density of the medium, and electrical energy is kinetic. There are thus two cases to be discussed.

I. *Magnetic Energy Kinetic.*—μ the density of the medium.

1. Magnetic force $= \mathbf{H} = \mu^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(XYZ)^{-\frac{1}{2}}] = RT^{-1} = YT^{-1}$, since R must now coincide with Y. This is the linear velocity directed along Y the magnetic axis.

2. Magnetic induction $= \mathbf{B} =$ Intensity of magnetization, $\mathbf{I} = \mu\mathbf{H} = \left(\frac{M}{XYZ}\right)\frac{Y}{T} =$ Linear momentum per unit volume.

3. Magnetic moment $= ml = \mathbf{I}(XYZ) = MYT^{-1} =$ Linear momentum.

4. Current strength $= C = \mathbf{H}Y = Y^2T^{-1}$.

5. Current density $= \mathbf{C} = C(YZ)^{-1} = YZ^{-1}T^{-1} =$ Angular velocity. The velocity is instantaneously directed along Y, the magnetic axis, and the axis of rotation is X, the axis of the electrical displacement. If the angular velocity be that of a vortex filament coinciding with the current, the strength of the current, C, becomes the strength of the vortex—product of angular velocity into cross-section of filament.

6. Electric displacement $= \mathbf{D} = CT = YZ^{-1} =$ an angular displacement. [According to the elastic solid theory, $\mathbf{D}$ would be of the dimensions of a *shear*, the planes XY being displaced parallel to each other in the direction Y.]

7. Electrical force $= \mathbf{E}$. Since $[\mathbf{ED}] = MY(XZ)^{-1}T^{-2}$, and $[\mathbf{D}] = YZ^{-1}$, $\mathbf{E} = MX^{-1}T^{-2}$. Taking $\mathbf{D}$ as an angular displacement, $\mathbf{E}$ must be of the dimensions of a *torque*, for $\mathbf{ED}$ is of the dimensions of work done per unit volume. [Let a cylinder rotate in a resisting medium. The resistance to the rotation is a tangential force per unit surface of the cylinder, directed everywhere parallel to the motion. If the axis of the cylinder be X (the electrical axis), and the radius Z, an element of its surface is of the dimensions YZ. Hence, the

resisting stress is of the dimensions $MX^{-1}T^{-2} = [\mathbf{E}]$. Thus **E** is a tangential force per unit area. Now **E** has a moment round X. Hence we have

$$[\mathbf{E}] = \left(\frac{MYZT^{-2}}{XYZ}\right),$$

indicating that the resisting stress over an element of surface is inversely as the distance of the element from the axis of rotation, and its moment directly as that distance.]

8. Voltage.—Writing $[\mathbf{E}] = \frac{MYZT^{-2}}{XYZ}$, we get

$$[E] = \left(\frac{MYZT^{-2}}{YZ}\right).$$

Now E, the line integral of **E**, is the source of the disturbance in a closed electrical circuit. Hence, taking the above mechanical illustration, E becomes the terminal torque required to maintain the motion of the cylinder against the resistance. "The terminal torque corresponds to the impressed voltage. It should be so distributed over the end B" (in this case a rotating tube) "that the applied force there is a circular tangential traction varying inversely as the distance from the axis" (Mr. Oliver Heaviside, Elec. Jan. 23, 1891). In the above case, an element of the tangent at a point is Y, and of the radius Z.

9. Vector potential $= \mathbf{A} = \mathbf{E}T = MX^{-1}T^{-1} = \frac{MYZT^{-1}}{XYZ}$ = angular momentum per unit volume.

10. Specific resistance.

$$\rho = \frac{\mathbf{E}}{\mathbf{C}} = \frac{MX^{-1}T^{-2}}{YZ^{-1}T^{-1}} = MZ(XY)^{-1}T^{-1}$$

= coefficient of viscosity = tangential force per unit area ÷ shear per unit time.

11. Self-induction.—Let a linear conductor carry a current C. Its self-induction is given by $L = \frac{p}{C}$, or dimensionally MY^{-2}. If **L** be defined as the self-induction of the conductor for a current of unit density,

$$\mathbf{L} = \frac{p}{\mathbf{C}} = MT^{-1}(YZ^{-1}T^{-1})^{-1} = MZY^{-1} = \frac{MZ}{YZ}$$

= moment of inertia per unit area; or $= \left(\frac{M}{YZ}\right)Z^2$ = moment of inertia of a disk of unit mass per unit area.

12. Specific inductive capacity $= k = [MZ(XY)^{-1}T^{-2}]^{-1}$. On the elastic solid theory k^{-1} would be defined as the rigidity of the medium. But $k^{-1} = RT^{-1}$, where R is specific resistance. Thus the interpretations of k and R go together. If R (as above) be a coefficient of viscosity, k^{-1} becomes "a *quasi* rigidity" of the medium "arising from elastic resistance to absolute rotation." (Mr. Oliver Heaviside, Elec. Jan. 23, 1891.)

13. Electrical charge $= q = \mathbf{D}(YZ) = Y^2 =$ (Product of angular displacement into area).

14. Magnetic pole $= m = \frac{ml}{l}$ = Magnetic moment per unit length = Linear momentum per unit length.

The connexion between Vortex Motion and Electromagnetism is shown in Basset's 'Hydrodynamics,' from which the following are extracted (vol. i. chap. iv.):—

1. Vortex filament = Electrical current.

2. Velocity of liquid (linear velocity, components u, v, w) = Magnetic force (components α, β, γ).

3. Molecular rotation (angular velocity, components ξ, η, ζ) = Current density (components u, v, w).

4. Velocity potential due to vortex (ϕ) = Magnetic potential of current (Ω).

5. Doublet sheet = Magnetic shell.

6. Circulation (k) = Work done in moving a magnetic pole once round current.

7. Flux through vortex = Potential energy of magnetic shell.

8. The action of a vortex filament upon the surrounding liquid is determined by the quantities L, M, N. These correspond to the components F, G, H of electromagnetic momentum (**A**).

Thus, when μ is of the dimensions of density and magnetic energy kinetic, the interpretations of the dimensional formulæ of electromagnetic quantities are identical with those of the corresponding quantities in the case of vortex motion.

In the Phil. Mag. vol. xxxi. p. 149, Prof. J. J. Thomson develops a method of representing electromagnetic effects by means of tubes of electrostatic induction distributed throughout the field. The axes of these tubes coincide with the axes of electric displacement, and the tubes terminate normally upon the surfaces of charged bodies. By means of these tubes it is possible to picture the changes going on when electricity passes through electrolytes and conductors, and when changes are produced in the electromagnetic field. If, now, we suppose these tubes of electrostatic induction to be vortex filaments, this method of representing electromagnetic phenomena becomes identical with that indicated by the interpretations above given to the dimensional formulæ.

The two rotational theories of electromagnetism which have been hitherto put forward are discussed by Mr. Oliver Heaviside (Elec. Jan. 23, 1891). The first of these is that already deduced above from dimensional considerations, by supposing μ to be of the dimensions of density. Here the axis of rotation at any point is the axis of the electric displacement; the angular velocity is the current-density; the velocity is magnetic force; the *torque* called into play when the angular velocity of a vortex changes is the electric force; the terminal torque, the source of the disturbance, is the voltage; the relative angular displacements of the vortices when their velocities change is the electric displacement; the linear momentum of the irrotationally moving liquid is magnetic induction or intensity of magnetization; the angular momentum is the vector potential; the strength of a vortex is the strength of the current; specific resistance is a quantity of the nature of viscosity, and k^{-1} of the nature of rigidity.

In the other system things are inverted. Here k is the density of the medium, electrical energy is kinetic, and the axes of the vortices coincide with the directions of the magnetic displacements. The interpretations of the formulæ of the various quantities are as below.

II. *Electrical Energy Kinetic.*—k the density of the medium.

1. $\mathbf{E}=k^{-\frac{1}{2}}[M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})]$

$$=M^{-\frac{1}{2}}(XYZ)^{\frac{1}{2}}M^{\frac{1}{2}}RT^{-1}(X^{-\frac{1}{2}}Y^{-\frac{1}{2}}Z^{-\frac{1}{2}})=RT^{-1}=XT^{-1}$$

(since R here must coincide with X).

Thus, $\mathbf{E}$ is the velocity of the medium.

2. $E = \mathbf{E}X = X^2T^{-1}$. Magnetic current. (See below, its interpretation depends upon that of m and $\mathbf{B}$.)

3. $m = ET = X^2$.

4. $\mathbf{B} = m(XZ)^{-1} = XZ^{-1} =$ an angular displacement. Thus, $m =$ product of angular displacement into cross section.

5. $\mathbf{C}_m = \mathbf{B}T^{-1}XZ^{-1}T^{-1} =$ Angular velocity. Thus, E the strength of the magnetic current corresponds to the strength of a vortex, $\mathbf{C}_m$ being the density of the magnetic current.

6. $e = MT^{-1}$. This corresponds to magnetic pole in the previous case.

7. $\mathbf{D} = e(YZ)^{-1} = M(YZ)^{-1}T^{-1} = \dfrac{MXT^{-1}}{XYZ} =$ Linear momentum per unit volume.

8. $\mathbf{C} = \mathbf{D}T^{-1} = \left(\dfrac{MXT^{-2}}{XYZ}\right) =$ Force per unit volume.

9. $C = MT^{-2}$. This corresponds to *voltage* in the previous case.

10. $\mathbf{H} = C(Y^{-1}) = MY^{-1}T^{-2}$. Tangential force per unit area, a quantity of the nature of a torque, corresponding to **E** in the former case.

It is thus seen that this system is simply the inverse of the other, and it is unnecessary, therefore, to go into more detail.

To summarize, therefore, we may say that, of the eight possible systems previously mentioned, only two give rise to dimensions whose interpretations are intelligible, natural, and connected as a whole; and these interpretations accord with the two rotational theories of electromagnetism which have been hitherto put forward. Of the other six, it was shown that four necessitated, in some form or other, fluid theories of electricity or magnetism, and that the interpretation of the formulæ of μ and k and other quantities are difficult. These remarks also apply to the other two cases, in which electrification and magnetization became respectively *volume-strains*. All other cases were disposed of by excluding :—

(1) All values of μ leading to dimensions involving higher or lower (fractional) powers of the fundamental units than those encountered in the case of ordinary dynamical quantities.

(2) All values which while satisfying (1) rendered scalar magnitudes vectors, or vectors scalar.

(3) All values which while satisfying (2) rendered the direction of the instantaneous linear displacement at a point in an isotropic medium parallel to Z, and therefore normal to the plane of displacement (XY).

In the foregoing discussion I have attempted to generalize the ordinary dimensional expressions for physical quantities, and to carry somewhat further than he did the suggestion of Prof. S. P. Thompson that the idea of direction may be associated with the symbols as well as that of numerical magnitude. This is clearly possible—as the examples given show—in the case of ordinary dynamical quantities, and the method certainly enables us to distinguish between things which are physically different, though (if the ordinary system in which the idea of direction is suppressed be used) the dimensions are the same. The dimensional formula thus becomes for a physical quantity the analogue of a structural formula for a chemical compound.

Applying these ideas to the more difficult case of electrical and magnetic quantities, with the actual nature of which we are imperfectly acquainted, I have tried to use the method as an instrument for discriminating between probable and improbable hypotheses. It is at all events interesting to note that we are thus led to *two* possible dimensional systems which agree with the two principal groups of analogies by which electrical and magnetic facts have been illustrated by those who have most deeply studied these from the dynamical point of view. Between these two I do not attempt to decide ; but I cannot but hope that the methods I have suggested may make the study of dimensional formulæ not merely a convenient method of expressing numerical relations between fundamental and derived units, but a means of seeing more deeply into the physical facts they represent.

NOTE.—I regret that when I communicated the above paper to the Physical Society I was unaware of an important article by Prof. A. Lodge on "The Multiplication and Division of Concrete Quantities" ('Nature,' July 19, 1888). In this article Prof. Lodge discusses the general question of the products and quotients of concrete quantities, and of the meaning of physical relations between concretes of different kinds. He clearly points out that the dimensions of quantities do not always afford a test of their identity, and that in particular concretes of the order of angles and angular displacements are treated as pure numbers. The present paper may be regarded as an attempt to remove such difficulties by taking as fundamental in our theory of dimensions the vector instead of the cartesian unit length. By expressing the formulæ according to the conventions of vector algebra, we thus assign dimensions (in the extended meaning of the term) to all quantities which are physically recognized as *concrete*, the only quantities having unity as their dimensional formula being pure numbers and quantities of the nature of *tensors*, that is, ratios between similar and similarly directed concretes. Since, as pointed out by Prof. Lodge, the cause of the above difficulties lies in the neglect of the directional relations of quantities, these modified formulæ approximate more closely to what they are sometimes, and conveniently taken to be, namely *conventional expressions* of the *kinds* of different physical quantities.

INDEX.

A.

Page

A formula for calculating approximately the self-induction of a coil …… 15

A lecture experiment illustrating the effect of heat upon the magnetic susceptibility of nickel …… 47

Absorption-spectrum of cobalt glass produced by heat, on the changes in the …… 103

Additional notes on secondary batteries …… 44

Alternate current-condensers, on …… 49

—— current and potential difference analogies in the methods of measuring power …… 172

Alternating and experimental influence-machine, on an …… 125

Apparatus for measuring the compressibility of liquids, on an …… 147

Ayrton, W. E., and Mather, T., on the construction of non-inductive resistances …… 269

——, and Sumpner, W. E., on alternate current and potential difference analogies in the methods of measuring power …… 172

——, and Taylor, J. F., proof of the generality of certain formulæ published for a special case by Mr. Blakesley …… 114

B.

Baily, Walter, on the construction of a colour map …… 323

Batteries, additional notes on secondary …… 44

Beams, the influence of surface-loading on the flexure of …… 194

Bidwell, Shelford, a lecture experiment illustrating the effect of heat upon the magnetic susceptibility of nickel …… 47

——, some experiments with selenium cells …… 61

Blakesley, T. H., further contributions to dynamometry, or the measurement of power …… 106

——, on the solution of a geometrical problem in magnetism …… 56

Boys, C. V., on photographs of rapidly moving objects, and on the oscillating electric spark …… 1

——, and Watson, W., on the measurement of electromagnetic radiation …… 20

Breath figures …… 346

Burton, C. V., on a theory concerning the constitution of matter …… 275

C.

Page

Calculating approximately the self-induction of a coil, a formula for 15
Certain relations existing amongst the refractive indices of some of the chemical elements 44
Changes in the absorption-spectrum of cobalt glass produced by heat 103
Choking coils 319
Compressibility of liquids, on an apparatus for measuring the 147
Conroy, Sir John, on the changes in the absorption-spectrum of cobalt glass produced by heat 103
Constitution of matter, on a theory concerning the 275
Construction of a colour map 323
—— of non-inductive resistances 269
"Corresponding" temperatures, pressures, and volumes, on the generalizations of Van der Waals regarding 233
Croft, W. B., on breath figures 346
Current-condensers, on alternate 49

D.

Dale, Pelham, on certain relations existing amongst the refractive indices of some of the chemical elements 44
Dimensions of physical quantities to directions in space, on the relation of the 357
Dissociation into ions and its consequences, on the theory of 139
Drawing parabolic curves, on an instrument for 337
Dynamometry, or the measurement of power, further contributions to 106

E.

Edser, Edwin, and Stansfield, Herbert, on a portable instrument for measuring magnetic fields 338
Electric spark, on the oscillating 1
Electrolysis, on some points in 130
Electromagnetic radiation, on the measurement of 20
Electrometer as a wattmeter, on the 122
Electromotive forces of gold and of platinum cells 332
Expansion of chlorine by light as applied to the measurement of the intensity of rays of high refrangibility 185
Experiment illustrating the effect of heat upon the magnetic susceptibility of nickel 47
Experimental influence-machine, on an alternating and 125
Experiments in photo-electricity 67
—— with selenium cells, some 61

F.

Page

Fields of dynamos, some observations on the strength of the stray.. 338

FitzGerald, Maurice F., on flexure of long pillars under their own weight 315

Flexure of beams, the influence of surface-loading on the 194

—— of long pillars under their own weight 315

Forces of gold and of platinum cells, note on the electromotive 332

Formulæ published for a special case by Mr. Blakesley, proof of the generality of certain 114

Further contributions to dynamometry, or the measurement of power 106

G.

Generality of certain formulæ published for a special case by Mr. Blakesley, proof of the 114

Generalizations of Van der Waals regarding "corresponding" temperatures, pressures, and volumes 233

Geometrical problem in magnetism, the solution of a 56

Gladstone, J. H., and Hibbert, W., additional notes on secondary batteries 44

Glazebrook, R. T., on the value of some mercury resistance standards 159

H.

Herroun, E. F., on the electromotive forces of gold and of platinum cells 332

Hibbert, W., on a permanent magnetic field 306

——, and Gladstone, J. H., additional notes on secondary batteries. 44

High speeds, on a steam-engine indicator for 151

I.

Indicator for high speeds, on a steam-engine 151

Inductoscript, on the 353

Influence-machine, on an alternating and experimental 125

Influence of surface-loading on the flexure of beams, on the 194

Instrument for drawing parabolic curves 337

—— for measuring magnetic fields, on a portable 338

Intensity of rays of high refrangibility, the expansion of chlorine by light as applied to the measurement of the 185

Internal resistance of cells, note on the measurement of the 342

Inwards, Richard, on an instrument for drawing parabolic curves .. 337

Ionic velocity, on Kohlrausch's theory of 149

Ions, the theory of dissociation into 139

K.

Page

Kohlrausch's theory of ionic velocity, on........................ 149

L.

Lateral loads, struts and tie-rods with.......................... 290
Liquids, on an apparatus for measuring the compressibility of...... 147

M.

Magnetic field, on a permanent................................ 306
—— susceptibility of nickel, a lecture experiment illustrating the effect of heat upon the 47
Magnetism, the solution of a geometrical problem in 56
Matter, on a theory concerning the constitution of 275
Measurement of electromagnetic radiation, on the................ 20
—— of power, on the.. 106
—— of the intensity of rays of high refrangibility, the expansion of chlorine by light as applied to the 185
—— of the internal resistance of cells, note on the 342
Measuring magnetic fields, on a portable instrument for 338
—— the compressibility of liquids, on an apparatus for 147
Mercury resistance standards, on the value of some 159
Method of measuring power in transformers, on Mr. Blakesley's .. 164
Methods of measuring power, alternate current and potential difference analogies in the 172
Minchin, G. M., on experiments in photo-electricity 67

N.

Non-inductive resistances, on the construction of 269
Note on Kohlrausch's theory of ionic velocity 149
—— on the electromotive forces of gold and of platinum cells 332
—— on the measurement of the internal resistance of cells 342
Notes on photographs of rapidly-moving objects, and on the oscillating electric spark 1
—— on secondary batteries 44

O.

Observations on the strength of the stray fields of dynamos........ 338
Oscillating electric spark, notes on the.......................... 1

P.

Parabolic curves, on an instrument for drawing 337
Permanent magnetic field, on a................................ 306

Page

Perry, John, on choking coils 319
——, on a formula for calculating approximately the self-induction of a coil 15
——, on a steam-engine indicator for high speeds 151
——, on struts and tie-rods with lateral loads 290
——, on a table of zonal spherical harmonics, calculated by Messrs. C. E. Holland, P. R. Jones, and C. G. Lamb 221
——, on Mr. Blakesley's method of measuring power in transformers 164
Photoelectricity, experiments in 67
Photographs of rapidly-moving objects, notes on 1
Pickering, S. U., on the theory of dissociation into ions, and its consequences 139
Pillars under their own weight, flexure of long 315
Points in electrolysis, on some 130
Portable instrument for measuring magnetic fields, on a 338
Potential difference analogies in the methods of measuring power, alternate current and 172
Power in transformers, on Mr. Blakesley's method of measuring.... 164
——, on the measurement of........ 106
Problem in magnetism, the solution of a geometrical 56
Proof of the generality of certain formulæ published for a special case by Mr. Blakesley........ 114

R.

Radiation, on the measurement of electromagnetic 20
Rapidly-moving objects, notes on photographs of 1
Rays of high refrangibility, the expansion of chlorine by light as applied to the measurement of the intensity of 185
Relation of the dimensions of physical quantities to directions in space, on the 357
Relations existing amongst the refractive indices of some of the chemical elements, on certain 44
Resistance of cells, note on the measurement of the internal 342
Resistances, on the construction of non-inductive 269
Richardson, A., on the expansion of chlorine by light as applied to the measurement of the intensity of rays of high refrangibility . 185

S.

Secondary batteries, additional notes on 44
Selby, A. L., on the variation of surface-tension with temperature. . 119
Selenium cells, some experiments with 61
Self-induction of a coil, a formula for calculating approximately the. 15
Skinner, S., on an apparatus for measuring the compressibility of liquids 147

Page
Smith, E. Wythe, note on the measurement of the internal resistance of cells 342
Smith, F. J., on the inductoscript........ 353
Solution of a geometrical problem in magnetism, on the 56
Some experiments with selenium cells 61
—— points in electrolysis 130
Spherical harmonics, table of zonal 221
Stansfield, Herbert, and Edser, Edwin, on a portable instrument for measuring magnetic fields 338
Steam-engine indicator for high speeds, on a 151
Strength of the stray fields of dynamos, observations on the 338
Struts and tie-rods with lateral loads, on........ 290
Sumpner, W. E., and Ayrton, W. E., on alternate current and potential difference analogies in the methods of measuring power 172
Surface-loading on the flexure of beams, the influence of 194
Surface-tension with temperature, on the variation of 119
Swinburne, J., on alternate current-condensers 49
——, on some points in electrolysis 130
——, on the electrometer as a wattmeter 122

T.

Table of zonal spherical harmonics, calculated by Messrs. C. G. Holland, P. R. Jones, and C. G. Lamb 221
Taylor, J. F., and Ayrton, W. E., proof of the generality of certain formulæ published for a special case by Mr. Blakesley 114
Temperature, on the variation of surface-tension with 119
Temperatures, pressures, and volumes, on the generalizations of Van der Waals regarding “corresponding” 233
Theory concerning the constitution of matter 275
—— of dissociation into ions, and its consequences 139
Tie-rods with lateral loads, struts and........ 290
Transformers, on Mr. Blakesley's method of measuring power in.... 164

V.

Value of some mercury resistance standards, on the 159
Variation of surface-tension with temperature, on the 119

W.

Watson, W., and Boys, C. V., on the measurement of electromagnetic radiation 20
Wattmeter, on the electrometer as a 122
Whetham, W. C. D., on Kohlrausch's theory of ionic velocity 149

Page
Williams, W., on the relation of the dimensions of physical quantities to directions in space 357
Wilson, C. A. Carus, on the influence of surface-loading on the flexure of beams 194
Wimshurst, J., on an alternating and experimental influence-machine 125

Y

Young, Sydney, on the generalizations of Van der Waals regarding "corresponding" temperatures, pressures, and volumes 233

Z.

Zonal spherical harmonics, table of 221

Printed by TAYLOR and FRANCIS, Red Lion Court, Fleet Street.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1890–91.

February 7th, 1890.

Prof. A. W. **Reinold**, Past President, in the Chair.

The following were elected Members of the Society:—

Mr. E. W. **Smith** and Mr. C. E. **Holland**, B.A.

The following communication was completed:—

"On Galvanometers." By Prof. W. E. **Ayrton**, F.R.S., Mr. T. **Mather**, and Dr. W. E. **Sumpner**.

February 21st, 1890.

Prof. G. **Carey Foster**, Past President, in the Chair.

Mr. S. **Evershed** was elected a Member of the Society.

The following communications were made:—

"On a Carbon Deposit in a Blake Telephone Transmitter." By Mr. F. B. **Hawes**.

"The Geometrical Construction of Direct-reading Scales for Reflecting Galvanometers." By Mr. A. P. Trotter.

"A Parallel Motion suitable for Recording Instruments." By Mr. A. P. Trotter.

March 7th, 1890.

Prof. W. E. Ayrton, President, in the Chair.

The following were elected Members of the Society:—

Sir H. Mance, Mr. L. R. Shorter, Mr. C. Thompson, and Mr. A. D. Waller.

The following communications were made:—

"On Bertrand's Refractometer." By Prof. S. P. Thompson.

"An Apparatus for Distilling Mercury in a Vacuum." By Prof. Dunstan and Mr. Dymond.

"The Theory of Osmotic Pressure and its bearing on the Nature of Solution." By Mr. S. U. Pickering, M.A.

March 21st, 1890.

Prof. W. E. Ayrton, President, in the Chair.

Mr. A. E. Childs was elected a Member of the Society.

The following communications were made:—

"The Villari Critical Point in Nickel." By Mr. Herbert Tomlinson, F.R.S.

"On Bertrand's Idiocyclophanous Prism." By Prof. S. P. Thompson.

"On the Shape of Moveable Coils used in Electrical Measuring Instruments." By Mr. T. Mather.

April 18th, 1890.

Prof. W. E. **Ayrton**, President, in the Chair.

Mr. W. B. **Croft** was elected a Member of the Society.

The following communications were made:—

"On same recent Magnetic Work." By Prof. **Rücker**, F.R.S.

"A Theory of Permanent Magnetism." By M. M. F. **Osmond**.

May 2nd, 1890.

Prof. W. E. **Ayrton**, President, in the Chair.

The following communications were made:—

"The Distribution of Flow in a Strained Elastic Solid." By Prof. C. A. **Carus-Wilson**.

"On Photographs of Rapidly Moving Objects." By Prof. C. V. **Boys**, F.R.S.

"On the Oscillating Electric Spark." By Prof. C. V. **Boys**, F.R.S.

May 16th, 1890.

Prof. W. E. **Ayrton**, President, in the Chair.

Professor W. **Coleman** was elected a Member of the Society.

The following communications were made:—

"On Huyghens' Gearing in Illustration of Electric Induction." By Lord **Rayleigh**.

"On Dr. König's Researches on the Physical Basis of Music." By Prof. S. P. **Thompson**.

June 6th, 1890.

Prof. W. E. **Ayrton**, President, in the Chair.

The following communications were made:—

"The Effect of Change of Temperature on the Villari Critical Point of Iron." By Mr. H. **Tomlinson**, F.R.S.

"On the Diurnal Variation of the Magnet at Kew." By Mr. W. G. **Robson** and Mr. S. W. J. **Smith**.

June 20th, 1890.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"On the Stretching of Liquids." By Professor A. W. Worthington.

"On the Measurement of Electromagnetic Radiation." By Mr. C. V. Boys, Mr. A. E. Briscoe, and Mr. W. Watson.

"Notes on Secondary Batteries." By Dr. Gladstone, F.R.S., and Mr. Hibbert.

"An easy Rule for Calculating approximately the Self-Induction of a Coil." By Prof. J. Perry, F.R.S.

November 14th, 1890.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"On certain Relations existing among the Refractive Indices of the Chemical Elements." By the Rev. T. Pelham Dale, M.A.

"On Tables of Spherical Harmonics, with Examples of their Practical Use." By Prof. J. Perry, F.R.S.

November 28th, 1890.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"Additional Notes on Secondary Batteries." By Dr. Gladstone, F.R.S., and Mr. Hibbert.

"An Illustration of Ewing's Theory of Magnetism." By Prof. S. P. Thompson.

"The Solution of a Geometrical Problem in Magnetism." By Mr. T. H. Blakesley, M.A.

December 12th, 1890.

Prof. W. E. **Ayrton**, President, in the Chair.

The following communications were made:—

"Some Experiments with Selenium Cells." By Mr. **Shelford Bidwell**, F.R.S.

"On Alternate Current Condensers." By Mr. **James Swinburne**.

January 16th, 1891.

Prof. W. E. **Ayrton**, President, in the Chair.

The following communications were made:—

"On Photo-Electricity." By Prof. G. M. **Minchin**, M.A.

"A Lecture-Room Apparatus for determining the Acceleration due to Gravity." By Prof. F. R. **Barrell**, B.A.

Annual General Meeting.

February 13th, 1891.

Prof. A. W. **Reinold**, Past President, in the Chair.

The following Report of the Council was read by the Chairman:—

The Council have this year, as heretofore, to report a steady increase in the number of Members of the Society, which now reaches 365.

It is hoped that a perseverance in the present day and hour of Meeting, viz. *Alternate Fridays*, at 5 P.M., will ultimately result in a larger attendance at the ordinary Meetings. Since the last Annual General Meeting there have been 13 ordinary Meetings, the average attendance being 45. On one occasion, the 16th of May, when Prof. S. P. Thompson and M. König displayed to the Society the very beautiful Acoustic apparatus which the latter gentleman has carried to extreme perfection, the meeting was a thoroughly crowded one.

The translation of Van der Waals's memoir on the Continuity of the Liquid and Gaseous States of Matter, by Prof. Threlfall, Member of the Society, and Mr. J. F. Adair, has been completed and issued to Members of the Society.

Volta's Works.—The Council hope to issue the translation of Volta's works before the next Annual Meeting.

The loss of Members by death has been, the Council is glad to say, very small this year—Prof. Lant Carpenter being the only Member of the Society whose name occurs in the Obituary notice of the 'Times' at the end of 1890.

No loss has occurred in the ranks of the Honorary Members of the Society.

The following additions have been made to the Library:—

Newspapers and Magazines:—

Nature.
The Electrical Review.
Engineering.
The Electrician.
The Electrical Engineer.
Electrical Plant.
The Open Court.
Crónica Cientifica.
Ingeniero y Ferretero Español y Sud Americano.
The Philosophical Magazine.
Beiblätter der Physik.
Annalen der Physik.
Journal de Physique.

Journals of Societies, British:—

Proceedings of the Royal Society.
Journal of the Society of Arts.
Journal of the Institute of Electrical Engineers.
Proceedings of the Institute of Mechanical Engineers.
Quarterly Journal of the Royal Meteorological Society.
Proceedings and Transactions of the Royal Dublin Society.
Transactions of the Middlesex Natural History and Science Society.
Proceedings of the Philosophical Society of Glasgow.
Proceedings of the Philosophical Society of Cambridge.
Transactions of the Philosophical Society of Cambridge.

Memoirs of the Manchester Literary and Philosophical Society.
Proceedings of the Royal Institution.
Transactions of the Geological Society of Glasgow.
Proceedings of the Birmingham Philosophical Society.

Journals of Societies, Colonial :—

Journal and Proceedings of the Royal Society of New South Wales.
Proceedings of the Canadian Institute.
Transactions of the New Zealand Institute.
Magnetic and Meteorological Observations, Bombay.
Proceedings and Transactions of the Nova Scotian Institute of Natural Science.

Journals of Societies, Foreign :—

Proceedings of the Academy of Natural Sciences, Philadelphia.
Journal of the Physical and Chemical Society of Russia.
Proceedings of the Physical Society of Berlin.
Proceedings of the Physical Society of France.
Journal of the College of Science of Japan.
Transactions of the Seismological Society of Japan.

Books, &c. :—

Presented by the Astronomical Society :

Cavallo's Elements. 4 vols. 1803.
Electricity. Adams. 1784.
Electricity. Watson. 1746.
Electricity. Bennet. 1789.
Electricity. Ferguson. 1770. And a duplicate.
Electro-Chemistry. Singer. 1814.
Electricity. Priestly. 1778 & 1767.
Electricity. Cavallo. 2 vols. 1795.
Magnetism. Cavallo. 1795.
Galvanism. Wilkinson. 2 vols. 1804.
Electricity. B. Franklin. 1760, 1769.
Air-pump and Barometer. Brook. 1789.
Galvanism. Aldini. 1803.
Works of B. Franklin. 3 vols.
Natural Philosophy. Muschenbroek. 2 vols. 1744.
Magneto-Electric Light. Holmes.
Philosophical papers. B. Franklin. 1787.
Electricity. Viscount Mahon. 1779.
Magnetism and the Magnetic Needle. Lorimer. 1795.

Annual Report of Geological and Natural History Survey of Canada.

Report of the Ohio Meteorological Bureau.

On the Condensation of the Vapour of Water.

Catalogue of Books, Radcliffe Library.

Greenwich Magnetic and Meteorological Observations.

Greenwich Spectroscopic and Photographic Results.

Norwegian North-Sea Expedition. Actinida.

Dioptric Formulæ of Cylindrical Lenses. A Metric System of measuring and numbering Prisms and Lenses. Both by C. F. Prentice. New York.

Smithsonian Report, 1886 & 1887.

Annual Report of the Michigan Board of Agriculture.

A thick Octavo in Russian.

Decimal Coinage, Weights, and Measures.

On the Mathematical Theory of Light, by de Colnat d'Huart.

List of Canadian Hepaticæ.

Catalogue of Canadian Plants. Acrogens.

Scientific Papers of Clerk Maxwell. 2 vols.

Report of the Physical and Chemical Society of Russia on the Total Eclipse of the Sun of 1887.

Laplace's Works. Vol. 8.

On the Kerr Magnetic Effect, by Dr. R. Sissingh.

Abridgments of Specifications of Patent Office. 5 vols. By W. H Walenn.

W. H. SNELL, editor of the 'Electrician' newspaper, was born in the year 1858, and, dying on the 5th of March 1890, had but reached the early age of 31 when his friends were called upon to deplore his untimely end. Though he had been but a short time in the work which suited him so well, he certainly had made considerable progress in the useful but oftentimes desperate problem of making technical journalism scientific. Careful study of scientific literature and considerable experience in electrical work forbade him to allow dogma to outweigh science, and his sagacity enabled him to correlate unusual phenomena with the requirements of long and well established scientific principles.

The usefulness of such a mind would have been felt probably in any calling, but when, in 1886, the proprietors of the 'Electrician' newspaper appointed him Editor, he had his especial opportunity, and his efforts were steadily directed into converting that journal into a trustworthy record of the progress taking place in the branch of Physics with which it purported to deal. He became Member of this Society, April 28th, 1888.

Mr. **William Lant Carpenter** was born in the year 1841, and was a son of the distinguished physiologist, Dr. W. B. Carpenter, whom he accompanied in the exploration voyages of H.M.S. 'Porcupine' in the years 1869–70, being in charge of some of the chemical and physical investigations of that Expedition. For many years he was engaged in a manufacturing business at Bristol, where he took a part in the movement for promoting the foundation of the University College of that town. In the course of his travels he did his best by means of lectures to render the study of science popular in many quarters of the British Empire, and, as Gilchrist Lecturer, at home. In his latter years he was one of the managers of the School of Electrical Engineering in Hanover Square. He died, working hard at this post, on the 23rd of December, 1890.

He was also an active supporter of the Morley Memorial College, and of the movement for the Extension of University Teaching. He was elected a Member of this Society January 28th, 1882, and cooperated with our late President, Prof. Balfour Stewart, in some of his investigations on terrestrial magnetism.

The adoption of the Report was proposed and carried unanimously.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. W. E. Ayrton, F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. Gladstone, F.R.S.; Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; Sir Wm. Thomson, D.C.L., LL.D., P.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.

Vice-Presidents.—Dr. E. Atkinson; Walter Baily, M.A.; Prof. O. J. Lodge, D.Sc., F.R.S.; Prof. S. P. Thompson, D.Sc.

Secretaries.—Prof. J. Perry, D.Sc., F.R.S.; T. H. Blakesley, M.A., M.I.C.E.

Treasurer.—Prof. A. W. Rücker, M.A., F.R.S.

Demonstrator and Librarian.—C. Vernon Boys, F.R.S.

Other Members of Council.—Shelford Bidwell, M.A., LL.B., F.R.S.; W. H. Coffin; Major-General E. R. Festing, R.E., F.R.S.; Prof. G. F. Fitzgerald, M.A., F.R.S.; Prof. J. V. Jones, M.A.; Rev. F. J. Smith, M.A.; Prof. W. Stroud, D.Sc.; H. Tomlinson, B.A., F.R.S.; G. M. Whipple, B.Sc.; James Wimshurst.

Votes of thanks were passed to the Lords Committee of the Council on Education; to the Officers; and to the Auditors.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1890, TO DECEMBER 31ST, 1890.

Dr.	£	s.	d.	£	s.	d.
Balance in Bank, January 1, 1890				211	14	0
Entrance-Fees	15	0	0			
Subscriptions for 1887	1	0	0			
„ 1888	4	0	0			
„ 1889	16	0	0			
„ 1890	127	0	0			
„ 1891	2	0	0			
Excess Subscriptions		4	0			
Ten Life Subscriptions	100	0	0			
				265	4	0
One year's Dividend on £400 Furness 4 per cent. Debenture Stock, less Income Tax	15	12	0			
One year's Dividend on £575 Midland 4 per cent. Preference Stock, less Income Tax	22	8	6			
One year's Dividend on £200 Lancaster Corporation Stock, less Income Tax	7	16	0			
Three Quarters of a year's Dividend on £200 Metropolitan Board of Works Stock, less Income Tax	5	2	6			
One year's Dividend on £254$\frac{3}{8}$ New South Wales 3½ per cent. Stock	8	13	6			
				59	12	6
Received from Taylor and Francis:—						
Sales in 1890				50	13	0
Balance due to Treasurer				9	18	10
				£597	2	4

Cr.	£	s.	d.	£	s.	d.
Balance due to Treasurer				7	5	6
Excess Subscriptions returned					4	0
Taylor and Francis:—						
Proceedings, vol. x. (parts 2 & 3)	105	17	0			
Circulars, addressing and posting	7	17	0			
Members' separate copies	19	4	0			
Miscellaneous printing	25	9	0			
Physical Memoirs	72	9	9			
				230	16	9
H. and W. Brown				1	3	9
Dr. Lawson				8	10	8
Williams and Norgate				3	9	0
Haggard (Binding)				4	14	7
Gyde (Binding)				1	18	0
Attendance at Meetings and incidental expenses				7	10	3
Reporting				19	6	8
Bank charges					4	0
Petty Cash:—						
Professor Rücker				1	0	0
Professor Perry					19	4
Balance in Bank				309	19	10
				£597	2	4

Audited and found correct,

GEORGE FULLER, } *Auditors.*
A. H. FISON,

February 10th, 1891.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1890.

Assets.	£	s.	d.
Balance in Bank	309	19	10
Subscriptions due	194	0	0
£400 Furness 4 per cent. Debenture Stock	456	0	0
£575 Midland 4 per cent. Preference Stock	650	0	0
£200 Metropolitan Board of Works Stock	217	0	0
£200 Lancaster Corporation Stock	224	0	0
£254¾ New South Wales 3½ per cent. Stock	249	7	5
One Quarter's Dividend on £200 Metropolitan Board of Works Stock	1	14	2
	£2302	1	5

Liabilities.	£	s.	d.
Balance due to Treasurer	9	18	10
Subscriptions anticipated	4	0	0
Balance	2288	2	7
	£2302	1	5

Audited and found correct,

A. H. FISON,
GEORGE FULLER, } *Auditors.*

February 10th, 1891.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1891-92.

February 13th, 1891.

Prof. A. W. **Reinold**, Past President, in the Chair.

The following were elected Members of the Society:—

Mr. W. **Thorp**, Mr. G. W. **Yule**, and Mr. S. **Joyce**.

The following communication was made:—

"On the Change in the Absorption-Spectrum of Cobalt Glass produced by Heat." By Sir **John Conroy**, Bart., M.A.

Prof. **Minchin** showed some experiments in illustration of his paper on "Photo-electricity."

February 27th, 1891.

Prof. W. E. **Ayrton**, President, in the Chair.

Prof. A. **Gray**, M.A., was elected a Member of the Society.

The following communications were made:—

"Proof of the Generality of Certain Formulæ published for a

Special Case by Mr. Blakesley, with Tests of a Transformer." By Prof. W. E. Ayrton, F.R.S., and Mr. J. F. Taylor.

"Further Contributions to Dynamometry." By Mr. T. H. Blakesley, M.A.

March 6th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

The following were elected Members of the Society:—

Mr. H. A. Miers, M.A., and Mr. W. Lucas, M.A.

The following communication was made:—

"Note on Electrostatic Wattmeters." By Mr. J. Swinburne.

Prof. S. P. Thompson having taken the Chair,

The following communication was made:—

"Interference with Alternating Currents." By Prof. W. E. Ayrton, F.R.S., and Dr. W. E. Sumpner.

March 20th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"The Theory of Dissociation into Ions, and its Consequences." By Mr. S. U. Pickering, F.R.S.

"Magnetic Proof Pieces and Proof Planes." By Prof. S. P. Thompson.

April 17th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"On a Property of Magnetic Shunts." By Prof. S. P. Thompson.

"An Alternating-Current Influence Machine." By Mr. James Wimshurst.

"On Erecting Prisms for the Optical Lantern." By Prof. S. P. Thompson.

May 9th, 1891.

On this occasion the usual meeting gave place to a visit to Cambridge University. At a meeting in the Cavendish Laboratory,

Prof. W. E. Ayrton, President, in the Chair.

the following communications were made:—

"Some Experiments on the Electric Discharge in Vacuum Tubes." By Prof. J. J. Thomson, M.A., F.R.S.

"Some Experiments on the Velocity of Ions." By Mr. W. C. D. Whetham, B.A.

"On the Resistance of some Mercury Standards." By Mr. R. T. Glazebrook, M.A., F.R.S.

"On an Apparatus for Measuring the Compressibility of Liquids." By Mr. S. Skinner, M.A.

"Some Measurements with a Pneumatic Bridge." By Mr. W. N. Shaw, M.A.

May 22nd, 1891.

Prof. W. E. Ayrton, President, in the Chair.

Mr. F. H. Nevill was elected a Member of the Society.

The following communications were made:—

"On Dr. Schobben's Form of Lantern Stereoscope." By Mr. C. J. Woodward.

"On a new Form of Steam-Engine Indicator." By Prof. J. Perry, F.R.S.

"On Mr. Blakesley's Method of Measuring Power in Transformers." By Prof. J. Perry, F.R.S.

June 12th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

Mr. W. H. Dines, B.A., was elected a Member of the Society.

The following communications were made:—

"Alternate Current and Potential Difference Analogies in the Methods of Measuring Power." By Prof. W. E. Ayrton, F.R.S., and Dr. Sumpner.

"On a Clock for Pointing out the Direction of the Earth's Orbital Motion." By Prof. O. Lodge, F.R.S.

"Some Experiments with Leyden Jars." By Prof. O. Lodge, F.R.S.

June 26th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

Mr. J. Enright, B.Sc., was elected a Member of the Society.

The following communications were made:—

"The Construction of Non-inductive Resistances." By Prof. W. E. Ayrton, F.R.S., and Mr. T. Mather.

"On the Influence of Surface-Loading on the Flexure of Beams." By Prof. C. A. Carus-Wilson.

"On Pocket Electrometers." By Prof. C. V. Boys, F.R.S.

"On Electrification due to the Contact of Gases with Liquids." By Mr. J. Enright.

"On the Expansion of Chlorine by Heat." By Dr. Arthur Richardson.

November 6th, 1891.

Dr. E. Atkinson, Vice-President, in the Chair.

The following were elected Members of the Society:—

Miss Alice Lee, Mr. W. A. Shenstone, and Mr. F. McLean.

The following communication was made:—

"On the Generalizations of Van der Waals regarding Corresponding Temperatures, Pressures, and Volumes." By Prof. Sydney Young, D.Sc.

November 20th, 1891.

Prof. W. E. Ayrton, President, in the Chair.

The following communications were made:—

"On the Generalizations of Van der Waals." By Dr. Philippe Guye.

"A new Theory concerning the Constitution of Matter." By Dr. C. V. Burton.

December 4th, 1891.

Prof. W. E. AYRTON, President, in the Chair.

The following were elected Members of the Society:—

Messrs. P. L. GREY, A. ANDERSON, H. DAVEY, L. W. FULCHER,
H. H. HOFFERT, and W. WATSON.

The following communications were made:—

"A Permanent Magnetic Field." By Mr. W. HIBBERT.

"Note on Rotatory Currents." By Prof. W. E. AYRTON, F.R.S.

"On Struts and Tie-Rods laterally loaded." By Prof. J. PERRY, F.R.S.

December 18th, 1891.

Prof. W. E. AYRTON, President, in the Chair.

Mr. R. W. MOND was elected a Member of the Society.

The following communication was made:—

"Note on Interference with Alternating Currents." By Mr. H. KILGOUR.

January 22nd, 1892.

Prof. O. LODGE, Vice-President, in the Chair.

The following were elected Members of the Society:—

Mr. J. B. PEACE and Mr. E. G. HIGHFIELD.

The following communication was made:—

"On the Driving of Electromagnetic Vibrations and Electromagnetic and Electrostatic Engines." By Prof. G. F. FITZGERALD, F.R.S.

Annual General Meeting.

February 12th, 1892.

Prof. W. E. AYRTON, President, in the Chair.

The following Report of the Council was read by the Chairman:—

In the year which has elapsed since the last Annual General Meeting of this Society, there have been held 13 Ordinary Meetings in the Laboratory at Kensington, and one Special Meeting at the University of Cambridge. Thus there has been one meeting more than the usual number in one year; and it is the hope of the Council that with an increasing number of communications it will be found possible to multiply in proportion the number of meetings. Many difficulties, however, would have to be overcome before any definite increase—such as weekly meetings instead of only two in a month—can be safely adopted. It is unfortunately the case that Members are in the habit of delaying their communications till the Session is far advanced and the recess is approached. Such a course results in a block of business in June, so that the papers and subjects often fail to receive adequate attention, and at the end of the Session have even to be taken as read.

In the early portion of the Session, on the other hand, there is not unfrequently a smaller supply of communications than is sufficient for the number of meetings held at present.

Again, time could be saved towards the end of the Session by grouping papers of a kindred nature, and discussing them together. If Members would give the Secretaries long notice that they have certain papers in hand, something might be done in this direction; but with the present procedure it is impossible to reap any advantage from this idea.

The special meeting at Cambridge took place on May 9th, 1891. On that occasion the Members of the Society were hospitably entertained by the Colleges of Emmanuel and Trinity, and had the opportunity of inspecting the Cavendish Laboratory and of hearing papers from Members and others resident at the University. The Council feel that the Society has every reason to congratulate itself upon the success of this visit, and think that such visits might be made more frequently.

The Council has to record the deaths of the following Members of the Society:—

Prof. W. WEBER, Honorary Member; Dr. W. H. STONE;
W. G. GREGORY, M.A.; Prof. J. C. ADAMS, F.R.S.

The Council recommend the name of Professor Van der Waals for election as Honorary Member to fill the vacancy caused by the death of Prof. Weber.

The numbers of the Society now exceed 400.

The following additions have been made to the Library during the year 1891–92:—

Newspapers and Magazines:—

Nature.
The Electrical Review.
Engineering.
The Electrician.
Electrical Plant.
Ingeniero y Ferretero Español y Sud Americano.
Crónica Cientifica.
The Open Court.
The Philosophical Magazine.
Beiblätter der Physik.
Annalen der Physik.
Journal de Physique.
Annales de Chemie et de Physique.
Archives Générales de Médicine.

Journals of Societies, British:—

Proceedings of the Royal Society.
Journal of the Society of Arts.
Journal of the Institute of Electrical Engineers.
Proceedings of the Royal Institution.
Transactions of the Philosophical Society of Cambridge.
Proceedings of the Philosophical Society of Cambridge.
Proceedings of the Manchester Literary and Philosophical Society.
Transactions and Proceedings of the Royal Dublin Society.
Quarterly Journal of the Royal Meteorological Society.
Proceedings of the Institute of Mechanical Engineers.

Journals of Societies, Colonial:—

Journal and Proceedings of the Royal Society of New South Wales.
Mathematical and Physical Society of Toronto University, Papers.

Proceedings and Transactions of the Nova Scotia Institute of Natural Science.

Journals of Societies, Foreign:—

Proceedings of the Academy of Natural Sciences, Philadelphia.
Journal of the Physical and Chemical Society of Russia.
Proceedings of the Physical Society of Berlin.
Physical Society of France. Mémoires.
Physical Society of France. Séances.
Journal of the College of Science of Japan.
Bulletin International de l'Acad. Sci. Cracowie. Séances.

Books, &c.:—

Ohio Meteorological Report.
Geological and Natural History Survey of Canada, vol. iv.
Hong Kong Observatory, Observations.
Metric Measures. Latimer Clark.
North Atlantic Expedition. xx.
The State Weather Service. F. E. Nipher.
St. Louis Engineering Club. President's Address.
Smithsonian Report, 1888.
Studies in Statistics. G. B. Longstaff.
Traité pratique de la Thermométrie de Precision. C. E. Guillaume.
Documents relating to the fixing of a Standard of Time. Ottawa.
Recherches Expérimental Acrodynamiques et données d'Expérience. S. P. Langley.
Three Editorials on Tubercle. Philadelphia.
Wissenschaftliche und Technische Arbeiten. Werner Siemens.
On the Causes of the Phenomena of Terrestrial Magnetism (in English and French), by Henry Wilde.
Catalogue of Books, Radcliffe Library, Oxford, added 1890.
*The Working and Management of an English Railway. G. Findlay.
*The Practical Telegraph Handbook, by S. Poole.
*The First Book of Electricity and Magnetism. W. Perrin Maycock.
Astronomy and Astro-Physics.
Comparative Photographic Spectra of the Sun and Metals. F. McClean.

---

* Presented by Messrs. Whittaker & Co.

Prof. WILHELM EDUARD WEBER, who died in June of last year, at the age of 87, was one of the first of the Honorary Members elected by the Society. It was fitting that this should be so, for though advancing years prevented his taking an active part in the most recent progress of Physics, he was one of those who had the chief share in laying the foundations on which modern conceptions in one of the most important branches of our science rest. He was not, like Faraday, a discoverer of new phenomena by which whole new territories were added to the domain of science; his work was rather that of systematizing, ascertaining the exact mutual bearings of phenomena, and establishing permanent landmarks whereby all after-comers might guide their ways and pass swiftly and easily over what was before a confused and difficult region.

In the Royal Society Catalogue more than sixty papers on various branches of Physics are enumerated of which W. E. Weber was the author. About twenty of these, dating from 1825 to 1835, relate, with very few exceptions, to questions of acoustics; the remainder are concerned with magnetism and electricity. There can be no doubt that the direction which Weber's scientific activity took from about 1835 onwards was determined by his having been, as Professor of Physics at Göttingen, the colleague and associate of Gauss. For seven years (1836 to 1842) they cooperated in directing the work of the *Magnetische Verein* and in issuing the important series of volumes embodying the '*Resultate aus den Beobachtungen*' of this association.

The great work of Weber's life was the establishment on a firm basis of the *absolute* system of measurement for electrical magnitudes. His first step in this direction seems to have been the absolute measurement of the strength of an electric current in 1840. Weber's first definition of the unit strength of current, and the experimental method by which he measured actual currents in terms of this unit, were founded directly on Gauss's definition and method of measuring the absolute strength of a magnetic field. From the measurement of currents he went on to the measurement of electro-chemical equivalents, electromotive forces, and resistances. One very important piece of work was the verification of Ampère's law of the mutual action between two currents, and the exact determination of the equivalence between electric circuits and magnets.

As one result of this investigation, Weber showed how the definition and measurement of the strength of currents may be founded upon the observation of the forces exerted between two parts of the same current, without any reference to the unit of magnetic force. He also showed that a corresponding definition of the unit of electromotive force may be given, and consequently that electrical resistance may be measured in absolute units independently of the determination of the strength of a magnetic field. His last published work seems to have been a determination of absolute resistance, carried out in conjunction with F. Zöllner in 1880.

In connexion with recent investigations relating to the propagation of electrical vibrations, it is interesting to find Weber remarking, in 1846, that the laws of electrodynamic force would not unlikely be susceptible of simpler expression, if account were taken of an intervening medium, than is possible when the phenomena are considered from the point of view of action at a distance; and, further, that it is not improbable that the medium concerned in electric phenomena may be the luminiferous æther. Again, in a paper in 1864 on Electric Vibrations, he determined their velocity of propagation as $3{\cdot}107 \times 10^{10}$ centim. per second, and pointed out that this is the velocity of light, observing that Kirchhoff had previously come to the same result and drawn attention to the same coincidence.

With regard to Weber's experimental work, it may be said that no better models of careful and accurate work and clear statement of results obtained can to this day be put in the hands of a student of Physics than his and Gauss's papers in the *Resultate aus den Beobachtungen des magnetischen Vereins* and Weber's *Elektrodynamische Maassbestimmungen.*

Weber was born at Wittenberg in 1804; he studied at the University of Halle, where he became Professor-Extraordinary of Physics in 1828. He received a call to the Chair of Physics at Göttingen in 1831, but he was ejected from it, on political grounds, in 1837. From 1843 to 1849, when he returned to Göttingen, he was Professor of Physics at Leipzig. He had two brothers—Ernst Heinrich (born in 1795, died 1878), Professor of Anatomy and Physiology at Leipzig, in conjunction with whom he published, in 1825, the celebrated work on Wave-Motion (*Die Wellenlehre auf Experimente gegrundet*); and a younger brother, Eduard Friedrich, who was a distinguished anatomist, and was for many years prosector in the University of Leipzig. W. E. Weber was elected a Foreign Member of the Royal Society in 1850; his brother Ernst received the same honour in 1862.

Walter George Gregory (the youngest son of George J. G. and Eliza Gregory, Eglesfeld House, Dorchester, Dorset) was born on January 31st, 1859. He commenced his education at the Dorchester Grammar School, afterwards going to Weymouth College, and then to Bristol Grammar School, whence, in 1878, he obtained an open Mathematical Scholarship at Queen's College, Oxford. He took his B.A. degree in Mathematics in June 1882, obtaining two First Classes, and then worked for a year at the Clarendon Laboratory. On leaving Oxford, in 1883, he obtained the appointment of Demonstrator in Physics at the Royal Indian Engineering College, Coopers Hill, which appointment he retained till within two months of his death, which occurred on November 25, 1891.

Never very strong, the last years of his life were spent in a constant struggle against ill-health, which left him little superfluous energy to devote to work outside the ordinary routine. At the same time, it was a matter of surprise to those that knew him that he managed to do what he did. He was devoted to his subject, and practically lived in it; clear-headed in a remarkable degree, he could take in very quickly the conditions of a physical problem, and follow up its side issues, and consequently could often make most valuable suggestions to others in their work. He had a decided liking for mechanism and practical applications of scientific facts from his earliest years, and was constantly working at some invention or other of his own. He was an invaluable colleague in a Physical Laboratory in the designing of apparatus and doing anything requiring delicate manipulation. He contributed two papers to the Physical Society in November 1889, "On a Method of Driving Tuning-Forks Electrically," and "On a New Electric Radiation Meter," which furnish a slight indication of what he might have done with health and strength.

John Couch Adams was born at Lidcot, near Launceston, in Cornwall, on June 5, 1819. He received his early education at the village school and at Devonport, where he gave evidence of his remarkable faculty for mathematical and astronomical study. In October 1839 he entered at St. John's College, Cambridge; and in 1843 he graduated as Senior Wrangler and first Smith's Prizeman, becoming shortly afterwards a Fellow and tutor of his College.

Both before and after taking his degree he was fascinated by a problem which was at that time profoundly interesting astronomers —the irregularities shown by the planet Uranus in its motion. Its orbit differed from the elliptic path which an undisturbed planet would have pursued; and as the deviations could not be explained by the influence of the other known planets, it was supposed that there must be a more remote planet which had not been observed. To the search for this unknown planet Adams devoted all the energies of his mathematical genius, and everyone knows the brilliant success with which his labours were crowned. His solution was communicated to Prof. Challis in September 1845, and to the Astronomer-Royal in the following month. We need only refer to the facts that similar work was done in 1846 by Leverrier; that the French astronomer's results, unlike those of the English investigator, were at once made known; and that on September 23, 1846, the planet Neptune was found by Dr. Galle, of Berlin, on the basis of Leverrier's elements. Adams and Leverrier rank as joint discoverers, and, as such, they received on February 11, 1848, the gold medal of the Royal Astronomical Society. Some members of Adams's college, in order to mark their sense of the importance of his achievement, raised a fund, which the University accepted, for the founding of a prize, to be called "The Adams Prize," to be awarded every two years to the author of the best essay on some subject of pure mathematics, astronomy, or other branch of natural philosophy. In 1851 he was elected President of the Royal Astronomical Society.

As he did not take orders, his Fellowship at St. John's expired in 1852, but he continued to reside in the College until 1853, when he was elected to Pembroke. In 1858 he was appointed Professor of Mathematics at the University of St. Andrews, but he held this office only during a single session. He became the Lowndean Professor of Astronomy and Geometry, at Cambridge, in 1859, in succession to the late Prof. Peacock, and retained this position during the remainder of his life.

Meanwhile, he had been carrying on many important investigations; and, until ill-health disabled him, his labours were never seriously interrupted. Foremost among his later achievements were the results of his researches on the Moon and on the theory of the November meteors. In 1866 the Royal Astronomical Society awarded him its gold medal for his lunar researches. He had succeeded Prof. Challis as Director of the Cambridge Observatory in 1861; and in 1884 he served as one of the delegates for Great Britain at the International Meridian Conference at Washington.

For about a year and a half before his death Prof. Adams was too ill to do as much work as he had been accustomed to do, and during the last ten weeks he was confined to bed. He died on the morning of January 21.

He was a Fellow of the Royal Society, and of the leading foreign scientific bodies; and honorary degrees were conferred upon him by his own University and by Oxford. The post of Astronomer-Royal was offered to him by the First Lord of the Admiralty in 1881, on Sir George Airy's retirement, but declined by him on the ground of age.

The adoption of the Report was proposed and carried unanimously.

The Society then elected Professor Van der Waals an Honorary Member of the Society.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. G. F. FITZGERALD, M.A., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. GLADSTONE, F.R.S.; Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; Lord KELVIN, D.C.L., LL.D., P.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Prof. W. E. AYRTON, F.R.S.

Vice-Presidents.—Prof. A. W. RÜCKER, M.A., F.R.S.; WALTER BAILY, M.A.; Prof. O. J. LODGE, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.

Secretaries.—Prof. J. PERRY, D.Sc., F.R.S.; T. H. BLAKESLEY, M.A., M.Inst.C.E.

Treasurer.—Dr. E. ATKINSON.

Demonstrator and Librarian.—C. VERNON BOYS, F.R.S.

Other Members of Council.—SHELFORD BIDWELL, M.A., LL.B., F.R.S.; W. E. SUMPNER, D.Sc.; Major-General E. R. FESTING, R.E., F.R.S.; J. SWINBURNE; Prof. J. V. JONES, M.A.; Rev. F. J. SMITH, M.A.; Prof. W. STROUD, D.Sc.; L. FLETCHER, M.A., F.R.S.; G. M. WHIPPLE, D.Sc.; JAMES WIMSHURST.

Votes of thanks were passed to the Lords Committee of the Council on Education; to the OFFICERS; and to the AUDITORS.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1891, TO DECEMBER 31ST, 1891.

Dr.	£	s.	d.	£	s.	d.
Balance in Bank, January 1, 1891				309	19	10
Life Subscriptions	120	0	0			
Entrance-Fees	23	0	0			
Subscriptions for 1888	2	0	0			
„ 1889	5	0	0			
„ 1890	14	0	0			
„ 1891	109	0	0			
Excess Subscriptions	1	2	0			
Prepaid Subscriptions:—1892	7	0	0			
„ 1893	1	0	0			
				282	2	0
Paid to Treasurer for copy of Joule's papers				0	16	0
One year's Dividend on £400 Furness 4 per cent. Debenture Stock, less Income Tax	15	12	0			
One year's Dividend on £575 Midland 4 per cent. Preference Stock, less Income Tax	22	8	6			
One year's Dividend on £200 Lancaster Corporation Stock, less Income Tax	7	16	0			
Five Quarters of a year's Dividend on £200 Metropolitan Board of Works Stock, less Income Tax	8	10	10			
One year's Dividend on £254⅔ New South Wales 3½ per cent. Stock, less Income Tax	8	13	6			
				63	0	10
Taylor and Francis				32	14	7
Balance due to Treasurer				1	0	0
				£689	13	3

Cr.	£	s.	d.	£	s.	d.
Balance due to Treasurer				9	18	10
Excess Subscriptions returned				1	2	0
„ received in 1890				2	0	0
Taylor and Francis:—						
Proceedings, vol. ii. (part 1)	32	9	0			
Postages, &c.	4	4	6			
Members' separate copies	6	16	6			
Miscellaneous printing	18	18	5			
				62	8	5
Haggard (Binding)				5	0	5
Williams and Norgate				4	7	0
Gyde (Binding)				3	0	0
Attendance at Meetings				7	2	2
Reporting				21	4	0
Barth, Royalty on Van der Waals' "Continuity of the Gaseous and Liquid States"				5	0	0
Bank charges					1	1
Petty Cash:—						
Mr. Blakesley	1	18	0			
Professor Rücker	2	1	0			
				3	19	0
Cheques not paid in				12	0	0
Balance in Bank				552	10	4
				£689	13	3

Audited and found correct,

HENRY M. ELDER,
ALFRED H. FISON, } *Auditors.*

February 3rd, 1892

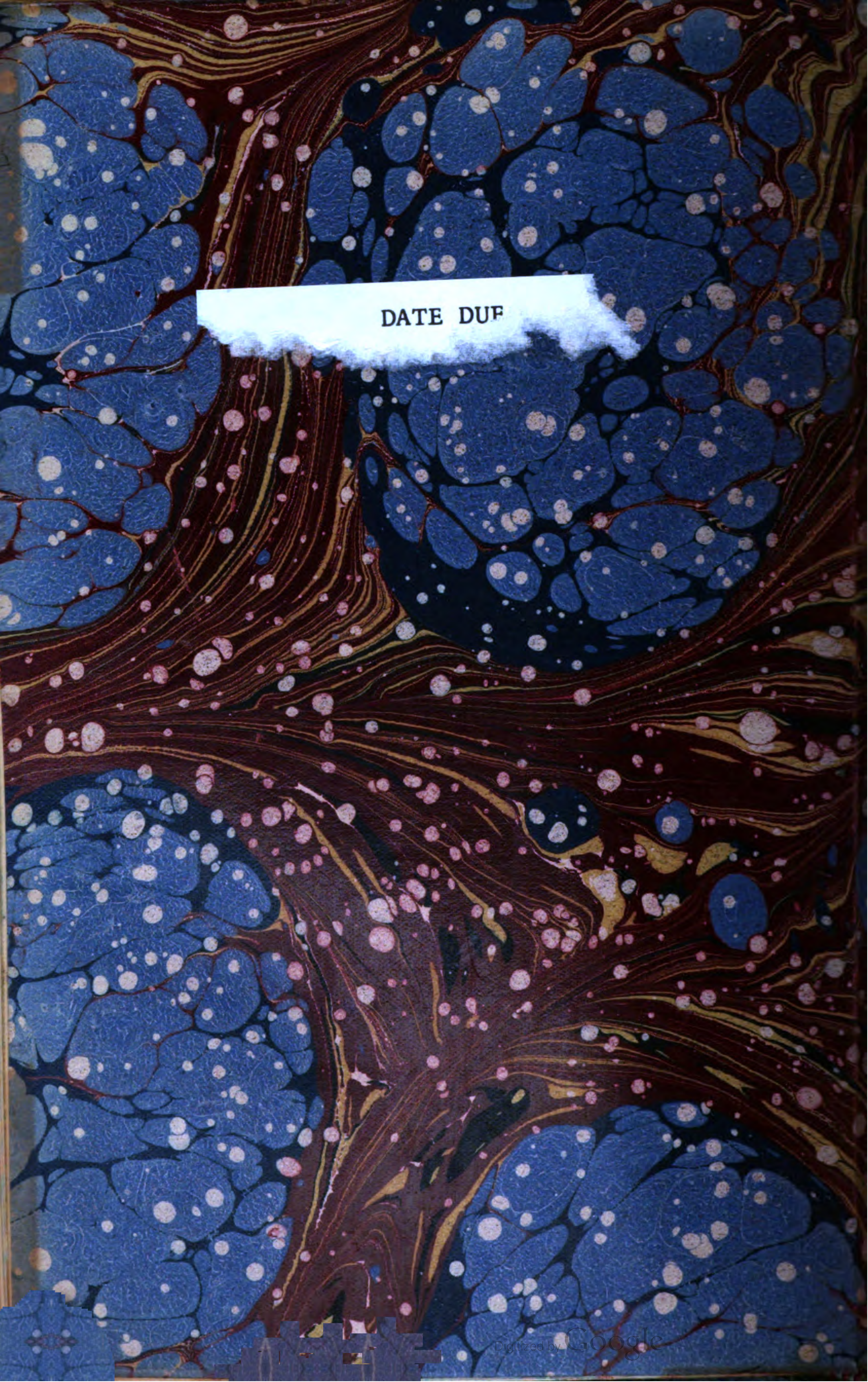
PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1891.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Balance in Bank	552	10	4	Balance due to Treasurer	1	0	0
Subscriptions due	180	0	0	Subscriptions anticipated	8	0	0
£400 Furness 4 per cent. Debenture Stock	456	0	0				
£575 Midland 4 per cent. Preference Stock	650	0	0				
£200 Metropolitan Board of Works Stock	217	0	0				
£200 Lancaster Corporation Stock	224	0	0				
£254⅜ New South Wales 3½ per cent. Stock	249	7	5				
Cheques unpaid	12	0	0	Balance	2531	17	9
	£2540	17	9		£2540	17	9

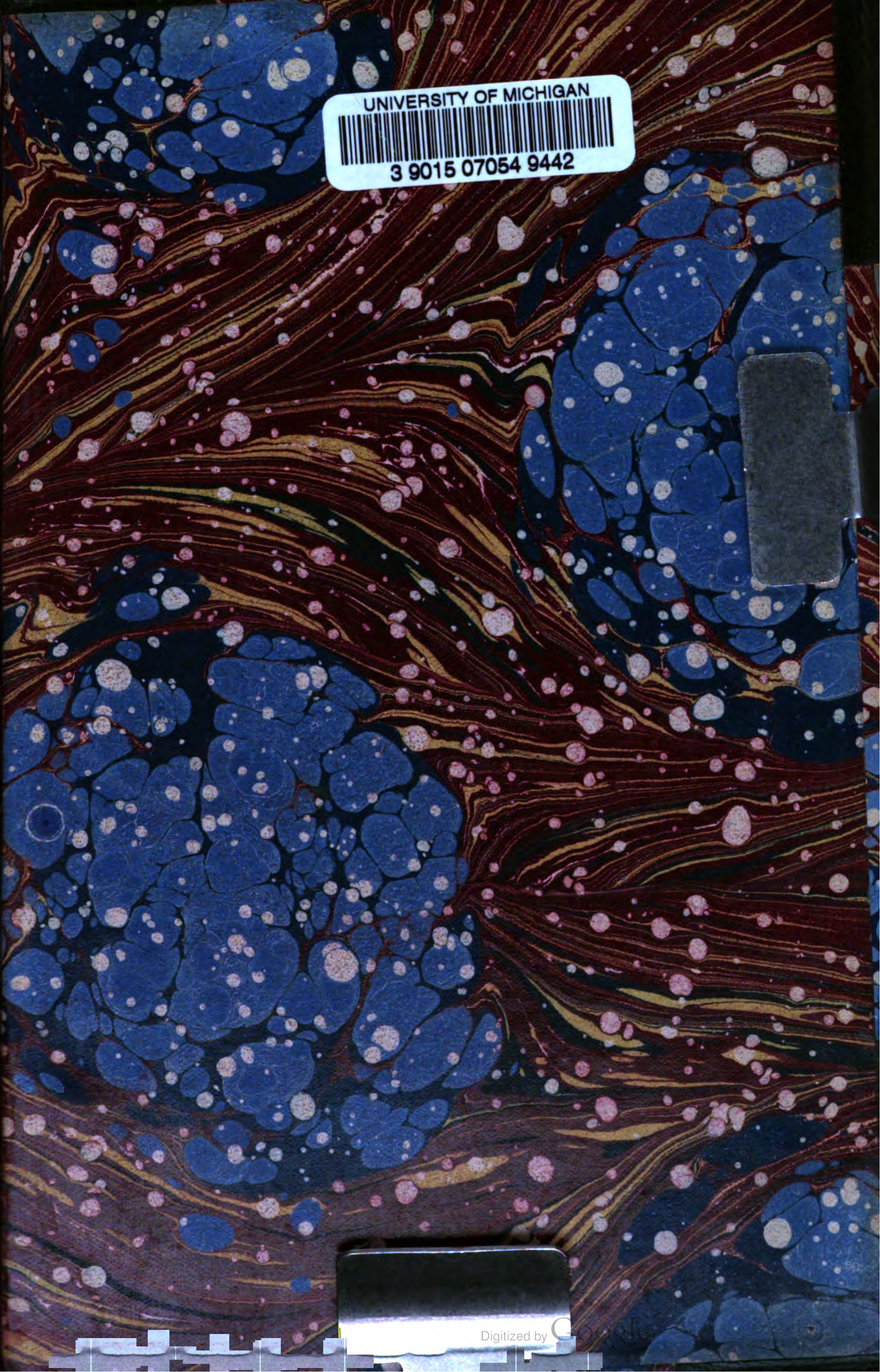
Audited and found correct,

A. H. FISON,
HENRY M. ELDER, } *Auditors.*

February 3rd, 1892.



DATE DUE



UNIVERSITY OF MICHIGAN
3 9015 07054 9442